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ITERATIVE METHODS FOR VARIATIONAL INEQUALITIES OVER THE SET OF COMMON FIXED POINTS OF NONEXPANSIVE SEMIGROUPS ON BANACH SPACES

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Introduction

Variational inequality theory was introduced by Hartman and Stampacchia (1966) as a tool for the study of partial differential equations with applications principally drawn from mechanics. Such variational inequalities were infinitely dimensional rather than finitely dimensional. The breakthrough in finite-dimensional theory occurred in 1980 when Dafermos recognized that the traffic network equilibrium conditions as stated by Smith (1979) had a structure of a variational inequality. This unveiled the methodology for the study of problems in economics, management science or operations research, and also in engineering, with a focus on transportation.

To-date problems which have been formulated and studied as variational inequality problems include: traffic network equilibrium problems, spatial price equilibrium problems, oligopolistic market equilibrium problems, financial equilibrium problems, migration equilibrium problems, as well as environmental network problems, and knowledge network problems. Variational inequality theory provides us with a tool for formulating a variety of equilibrium problems; it also allows to analyze qualitatively the problems in terms of existence and uniqueness of solutions, stability and sensitivity analysis, and it finally provide us with algorithms and their convergence analysis for computational purposes. It contains, as special cases, such well-known problems in mathematical programming as systems of nonlinear equations, optimization problems, complementarity problems, and fixed point problems.

Because of the important role of variational inequalities in mathematical theory as well as in many practical applications, it has always been a topical subject which attracts numerous researchers. Many mathematical methods and numerical algorithms for solving variational inequalities have been developed such as projection method by Lions (1977), auxiliary principle problem by Cohen (1980), proximal point method by Martinet (1970) and Rockafellar (1976); inertial proximal point method proposed by Alvarez and Attouch (2001), and Browder–Tikhonov regularization method (Browder, 1966; Tikhonov, 1963), etc. In Vietnam, in recent years the variational inequality problem has become an interesting and important topic for many groups of mathematical researchers major in Mathematical Analysis and Applied Mathematics. To name a few groups with publications on variational inequalities, we can cite: Buong and Thuy (Buong, 2012; Thuy, 2015), Yen (Lee et al., 2005; Tam et al., 2005), Muu and Anh (Anh et al., 2005, 2012), Sach (Tuan and Sach, 2004; Sach et al., 2008) and Khanh (Bao and Khanh, 2005, 2006), ... In addition, variational inequalities and some related problems such as fixed points and equilibrium problems have also been the topic of many young researchers and PhD students, for instance, Tuyen (2011, 2012), Duong (2011), Lang (2011, 2012), Duong (2011), Thong (2011), Phuong (2013), Thanh (2015), Khanh (2015) and Ha (2015), and others.

Let H be a Hilbert space with inner product $\langle ., . \rangle$. Let C be a nonempty closed and convex subset of H and let $F : H \to H$ be a mapping. The classical variational inequality, CVI(F, C) for short, is stated as follows:

$$\textit{Find an element } x_* \in C \textit{ such that } \langle F(x_*), x - x_* \rangle \geq 0, \quad \forall x \in C. \tag{0.1}$$

It has been known that the classical variational inequality CVI(F, C) is equivalent to the fixed point equation

$$x_* = P_C(I - \mu F)(x_*), \tag{0.2}$$

where P_C is the metric projection from H onto C, and $\mu > 0$ an arbitrary constant. When F is η-strongly monotone and L-Lipschitz continuous, the mapping $P_C(I - \mu F)$ in the right hand side of (0.2) is a contraction. Hence, the Banach contraction mapping principle guarantees that the Picard iteration based on (0.2) converges strongly to the unique solution of (0.1). Such a method is called the projection method. We remark that the fixed-point formulation (0.2) involves the projection P_C, which may not be easy to compute due to the complexity of the convex set C. In order to reduce the complexity probably caused by the projection P_C , Yamada (2001) introduced a hybrid steepest descent method for solving variational inequality (0.1) in a Hilbert space. His idea is using a nonexpansive mapping T whose fixed point set is the feasible set C, that is $C = \mathrm{Fix}(T)$, instead of the metric projection P_C, and a sequence $\{x_n\}$ is generated by the following algorithm:

$$x_{n+1} = Tx_n - \mu\lambda_{n+1}F(Tx_n), \quad n \geq 0, \tag{0.3}$$

with $\mu \in (0, 2\eta/L^2)$ and $\{\lambda_n\}_{n\geq 1} \subset (0, 1]$ satisfying some control conditions.

In this work, Yamada also considered the case when $C := \cap_{i=1}^N \mathrm{Fix}(T_i)$, the set of common fixed points of a finite family of nonexpansive mappings $(T_i)_{i=1}^N$, and proposed a cyclic iterative algorithm for solving variational inequality (0.1) over the feasible set $C := \cap_{i=1}^N \mathrm{Fix}(T_i)$. The strong convergence of the method is proved under an additional condition, namely an invariance property of the set of common fixed points of combinations of nonexpansive mappings in the family. Based on hybrid steepest descent method by Yamada, many authors have been considering methods for solving variational inequality over the feasible set C with more complicated structure such as the common fixed point set of countably infinite family of nonexpansive mappings (Yao et al., 2010; Wang, 2011) or nonexpansive semigroups which is the uncountably infinite family of nonexpansive mappings (Yang et al., 2012). These research works are important because they contain many applications arising from the theory of signal recovery problems, power control problems, bandwidth allocation problems and optimal control problems. In this thesis, we are interested in methods for solving variational inequalities over the set of common fixed points of nonexpansive semigroups $\{T(s) : s \geq 0\}$. This problem is linked with the evolution equation in the field of partial differential equations. Consider the differential equation $\frac{du}{dt} + Au(t) = 0$ which describes an evolution system where A is an accretive map from a Banach space E into itself. In Hilbert spaces, accretive operators are called monotone. At equilibrium state, $\frac{du}{dt} = 0$, and so a solution of $Au = 0$ describes the equilibrium or stable state of the system. This is very desirable in many applications, for example, in ecology, economics, physics, to name a few. Many studies showed that the solutions of an evolution equation with a m-accretive mapping $A : E \to E$ in a Banach space constitute a nonexpansive semigroup generated by operator A, and further, the set of common fixed points of $\{T(s) : s \geq 0\}$ is the set of zero points of A, that is $\mathcal{F} := \cap_{s \geq 0} \mathrm{Fix}(T(s)) = A^{-1}(0)$.

Along with the results achieved on different methods for solving variational inequality (0.1) in a Hilbert space H, many authors have recently studied solution methods for variational inequalities in Banach spaces. We know that, among Banach spaces, Hilbert space H is a space with very nice geometrical properties such as the parallelogram identity, or the existence of an inner product, the uniqueness of the projection onto a nonempty, closed and convex subset of H, etc. These properties make the study of the problem in Hilbert spaces much simpler than studying the problem in general Banach spaces. On the other hand, some methods for solving the problem converges in a Hilbert space but not necessarily in a general Banach space. This explains an important number of research works on extensions and generalizations recently appeared in the literature in the framework of Banach spaces. For some recent published results on solution methods for variational inequalities in Banach spaces, one needs to assume, in order to ensure their strong convergence, the weakly continuity of the normalized duality mapping. Until now it has been shown that the l^p, $1 < p < \infty$, satisfies this weakly continuity property while the $L^p[a, b]$, $1 < p < \infty$, does not. A natural question arising here is whether it is possible to develop methods for solving variational inequalities in Banach spaces without requiring the weakly continuity of the normalized duality mapping. If the answer is affirmative, then the scope of applications of the algorithms in question can be expanded towards more general Banach spaces such as $L^p[a, b]$, rather applicable only for l^p.

Another aspect of variational inequalities is that it is an ill-posed problem. To solve the class of these problems, we have to use stable methods, the so-called regularization methods. In practice, the input data are usually collected by observations or direct measurements. This means that there are errors on the input data, and the results received from the problem will not reliable enough; so it can lead to a wrong decision based on what we have considered as the solutions of the problem. These known facts yielded many interesting research publications for ill-posed problems including variational inequalities based on the Browder–Tikhonov regularization. In 2012, Buong and Phuong proposed a Browder–Tikhonov regularization method for problem of accretive variational inequalities over the set of common fixed points of countably infinite family of nonexpansive mappings $\{T_i\}_{i=1}^{\infty}$ in Banach spaces E using V-mapping as an improvement of W-mapping in some results of other authors.

Therefore, we can say that the variational inequality problem attracted numerous mathematicians, not only in Vietnam but also in the international community of researchers, to develop effective solution methods for solving this problem. The investigation of the problem in the framework of Banach spaces is a natural and necessary research topic to understand the problem in infinite dimension. For these reasons we chose a subject for this dissertation whose title is "*Iterative methods for variational inequalities over the set of common fixed points of nonexpansive semigroups on Banach spaces*". The main goal of this thesis is to study hybrid steepest methods and regularization methods for solving variational inequalities over the set of common fixed points of nonexpansive semigroups in Banach spaces. Specifically, the dissertation will address the following issues:

1. Devise implicit iterations based on hybrid steepest descent methods for accretive variational inequalities in uniformly convex Banach spaces without the use of sequentially weakly continuity property of the normalized duality mapping of Banach spaces.
2. Propose and analyze the corresponding explicit iterations of these implicit iterative methods for the same problem.
3. Suggest Browder–Tikhonov regularization methods for accretive variational inequalities and combine with inertial proximal point method to construct inertial proximal point regularization method for variational inequalities in uniformly convex and smooth Banach spaces; present another combination of the Browder–Tikhonov regularization

method with an explicit algorithm for variational inequalities in uniformly convex and q-uniformly smooth Banach spaces.

Besides the introduction, conclusion and references, the contents of the dissertation are presented in three chapters. In Chapter 1, we present some important preliminaries to prepare the presentation of the main results in the next chapters, specifically as some geometrical characteristics of Banach spaces, monotone type mappings, Lipschitz continuous mappings and variational inequalities in Banach spaces, like classical variational inequalities and some related problems, monotone variational inequalities and accretive variational inequalities. In Chapter 2, we introduce and analyze implicit iterative methods for accretive variational inequalities based on hybrid steepest descent methods in uniformly convex Banach spaces whose norm is uniformly Gâteaux differentiable. Also in this chapter we give the explicit versions of the corresponding implicit iterations for the same problem. In Chapter 3, we combine the Browder–Tikhonov regularization method with the inertial proximal point method to obtain the inertial proximal point regularization method for variational inequalities. We also combine the Browder–Tikhonov regularization method with an explicit iterative technique to devise an iterative regularization method for variational inequalities in uniformly smooth Banach spaces. We finally present some numerical results to illustrate the proposed methods at the end of Chapter 2 and Chapter 3.

Chapter 1
Preliminaries

Chapter 1 of the dissertation is devoted to introduce some basic preliminaries serving for the presentation of research results in the next chapters. Specifically, this chapter consists of 4 sections:

Section 1.1 is set up for the presentation of some geometrical characteristics of Banach spaces, definitions and some properties of monotone and accretive mappings, and Lipschitz continuous mapping.

In Section 1.2 we introduce nonexpansive semigroups and an application of nonexpansiveness for the Cauchy problem.

In Section 1.3, we give the statement of the problem of classical variational inequalities and some related problems such as system of equations, complementarity problem, optimization problem and fixed point problem.

In Section 1.4 we describe the problem of monotone and accretive inequalities in general Banach spaces. Also in this section we present the hybrid steepest descent method proposed by Yamada for solving a variational inequality over the set of common fixed points of a family of nonexpansive mappings.

Section 1.5 gives the statement of the problem of accretive variational inequalities over the feasible set that is the set of common fixed points of nonexpansive semigroups in Banach spaces. This problem is denoted $\mathrm{VI}^*(F, \mathcal{F})$ which will be considered throughout this dissertation.

Let $F : E \to E$ be an η-accretive and γ-pseudocontractive mapping with $\eta + \gamma > 1$. Let $\{T(t) : t \geq 0\}$ be a nonexpansive semigroup on E such that $\mathcal{F} := \cap_{s\geq 0}\mathrm{Fix}(T(s)) \neq \emptyset$, where $\mathcal{F}$ denotes the set of common fixed points of the nonexpansive semigroup $\{T(t) : t \geq 0\}$. We consider the problem:

$$\textit{Find } p_* \in \mathcal{F} \textit{ such that } \langle Fp_*, j(x - p_*)\rangle \geq 0 \ \forall x \in \mathcal{F}. \tag{1.1}$$

Proposition 1.1 *Let E be a real uniformly convex Banach space with a uniformly Gâteaux differentiable norm. Let $F : E \to E$ be an η-strongly accretive and γ-pseudocontractive mapping with $\eta, \gamma \in (0, 1)$ satisfying $\eta + \gamma > 1$. Let $\{T(s) : s \geq 0\}$ be a nonexpansive semigroup on E such that $\mathcal{F} := \cap_{s \geq 0} \mathrm{Fix}(T(s)) \neq \emptyset$. Then, the problem* (1.1) *has one and only one solution $p_* \in \mathcal{F}$.*

In the next chapters we will propose some methods for solving accretive variational inequalities based on hybrid steepest descent approach in uniformly convex Banach spaces having Gâteaux differentiable norm.

Chapter 2
Hybrid Steepest Descent Methods for Variational Inequalities over the Set of Common Fixed Points of Nonexpansive Semigroups

This chapter consists of three sections. In Section 2.1, we propose three implicit iterative schemes based on hybrid steepest descent method for variational inequalities $\mathrm{VI}^*(F, \mathcal{F})$ and in Section 2.2 we give the explicit versions of the methods studied in Section 2.1. A numerical example illustrating the proposed methods is presented and discussed in Section 2.3. Results of this chapter is taken from the articles (1) and (2) of the list of research papers published related to the dissertation.

2.1. Implicit Hybrid Steepest Descent Methods

2.1.1. State the Method

In this section we propose three implicit iterative methods based on the hybrid steepest descent method by Yamada for variational inequalities (1.1) in uniformly convex Banach spaces having uniformly Gâteaux differentiable norm. The first method is a convex combination of two mappings F_k and T_k defined, respectively, by $F_k x = (I - \lambda_k F)x$ and $T_k x = \frac{1}{t_k}\int_0^{t_k} T(s)x ds$, $x \in E$.

Method 2.1. *Start from an arbitrary point $x_1 \in E$, define $\{x_k\}$ by the following equation:*

$$x_k = \gamma_k F_k x_k + (1 - \gamma_k) T_k x_k, \ \ k \geq 1, \tag{2.1}$$

where $\gamma_k \in (0, 1), \lambda_k \in (0, 1]$ and $t_k > 0$ satisfy that $\lambda_k \to 0, t_k \to \infty$ as $k \to \infty$.

In the second methods, we do not use Bochner integral T_k but nonexpansive mapping $T(t_k)$ instead.

Method 2.2. *Start from an arbitrary point $x_1 \in E$, define $\{y_k\}$ by the following equation:*

$$y_k = \gamma_k F_k y_k + (1 - \gamma_k) T(t_k) y_k, \;\; k \geq 1, \tag{2.2}$$

where $\lambda_k \in (0, 1]$, $\gamma_k \in (0, 1)$ and $t_k > 0$ satisfy that $\lim_{k\to\infty} t_k = \lim_{k\to\infty} \frac{\gamma_k}{t_k} = 0$.

One might see that the structure of the two implicit iterative methods (2.1) and (2.2) is similar to each other but in the method (2.2), using direct mappings $T(t_k)$ with $t_k \to 0, k \to \infty$ without using Bochner integral, the method (2.2) is considered simpler to implement than the method (2.1). With the third method, by taking the composite of two mappings T_k and F_k, we construct an iterative sequence implicitly for variational inequalities $\mathrm{VI}^*(F, \mathcal{F})$ as follows.

Method 2.3. *Start from an arbitrary point $x_1 \in E$, define $\{w_k\}$ by the following equation:*

$$w_k = T_k F_k w_k, \;\; k \geq 1, \tag{2.3}$$

where $\lambda_k \in (0, 1]$ and $t_k > 0$ such that $\lambda_k \to 0$ and $t_k \to \infty$, as $k \to \infty$.

2.1.2. The Strong Convergence

Theorem 2.1 *Let E be a real uniformly convex Banach space with a uniformly Gâteaux differentiable norm. Let $F : E \to E$ be an η-strongly accretive and γ-pseudocontractive mapping with $\eta, \gamma \in (0, 1)$ satisfying $\eta + \gamma > 1$. Let $\{T(s) : s \geq 0\}$ be a nonexpansive semigroup on E such that $\mathcal{F} := \cap_{s\geq 0}\mathrm{Fix}(T(s)) \neq \emptyset$. Then, sequence $\{x_k\}$ defined by* (2.1) *converges strongly to p_*, the unique solution of variational inequality* (1.1) *as $k \to \infty$.*

Theorem 2.2 *Let E, F, $\{T(s) : s \geq 0\}$ and $\mathcal{F}$ be as in Theorem 2.1. Then, sequence $\{y_k\}$ defined by* (2.2) *converges strongly to p_*, the unique solution of variational inequality* (1.1) *as $k \to \infty$.*

Theorem 2.3 *Let E, F, $\{T(s) : s \geq 0\}$ and $\mathcal{F}$ be as in Theorem 2.1. Then, sequence $\{w_k\}$ defined by* (2.3) *converges strongly to p_*, the unique solution of variational inequality* (1.1) *as $k \to \infty$.*

Remark 2.1

(a) The proofs of convergence of the method (2.1) in Theorem 2.1, of the method (2.2) in Theorem 2.2 and of the method (2.3) in Theorem 2.3 do not require weakly continuity property of the normalized duality mapping of Banach spaces E.

(b) When $C = \mathcal{F} := \cap_{i=1}^{\infty}\mathrm{Fix}(T_i)$ is the set of common fixed points of countably infinite family of nonexpansive mappings, in 2013, Buong and Phuong proposed two implicit methods for solving (1.1) in a real uniformly convex Banach space with a uniformly Gâteaux differentiable norm. The first method has the same structure as (2.1) while the mapping T_k of (2.1) is replaced by V_k mapping.

(c) For some research results on the implicit iterative methods for the variational inequalities over the set of common fixed points of a family of nonexpansive mappings, we would like to mention those of Ceng et al. (2008), Chen and He (2007), Shioji and Takahashi (1998), Suzuki (2003), and Xu (2005). Ceng et al. (2008) also used Banach limit to prove the strong convergence of their methods.

2.2. Explicit Hybrid Steepest Descent Methods

2.2.1. State the Method

When constructing implicit iterative schemes in Section 2.2, a possible difficulty encountered by those methods in practice is the calculation of x_k at each iteration k. Indeed, we have to solve at each step an equation to find approximately x_k, and after a finite number of iterations we hope to obtain x_k closed to the exact solution of the interested problem. Stemming from the idea to overcome this issue of implicit iterative methods, we devise two explicit iterative methods based on (2.1) and (2.3).

Method 2.4. *Start from an initial guess $x_1 \in E$ arbitrarily, we* *generate $\{x_n\}$ explicitly as follows:*

$$x_{n+1} = \gamma_n F_n x_n + (1-\gamma_n) T_n x_n, \quad n \geq 1,\ x_1 \in E. \tag{2.4}$$

Method 2.5. *Start from an initial guess $x_1 \in E$ arbitrarily, we generate $\{x_n\}$ explicitly as follows:*

$$x_{n+1} = (1-\gamma_n)x_n + \gamma_n T_n F_n x_n. \tag{2.5}$$

Mappings T_n and F_n in (2.4) and (2.5) are defined respectively by

$$T_n x = \frac{1}{t_n}\int_0^{t_n} T(s)x ds, \tag{2.6}$$

$$F_n x = (I - \lambda_n F)x, \quad \text{for all } x \in E, \tag{2.7}$$

and $\{\gamma_n\}$, $\{\lambda_n\}$, $\{t_n\}$ satisfying the following conditions:

$$\lambda_n \in (0,1),\ \lambda_n \to 0,\ \sum_{n=1}^{\infty} \lambda_n = \infty, \tag{2.8}$$

$$\lim_{n\to\infty} t_n = \infty \text{ and } \{|t_{n+1} - t_n|\} \text{ is bounded} \tag{2.9}$$

$$\gamma_n \in (0,1) \text{ such that } 0 < \liminf_{n\to\infty} \gamma_n \leq \limsup_{n\to\infty} \gamma_n < 1. \tag{2.10}$$

2.2.2. The Strong Convergence

Proposition 2.1 *Let $F : E \to E$ be an η-strongly accretive and γ-strictly pseudocontractive mapping with $\eta+\gamma > 1$ and let $\{T(s) : s \geq 0\}$ be a nonexpansive semigroup on uniformly convex Banach space E having uniformly Gâteaux differentiable norm such that $\mathcal{F} = \cap_{s\geq 0}\mathrm{Fix}(T(s))$ is nonempty. Let $\{x_n\}$ be a bounded sequence such that $\lim_{n\to\infty}\|x_n - T(t)x_n\| = 0$ for all $t \geq 0$. Let also $p_* = \lim_{k\to\infty} y_k$ where $\{y_k\}$ is defined by (2.1) for all k, that is*

$$y_k = \gamma_k(I - \lambda_k F)y_k + (1-\gamma_k)T_k y_k$$

with $T_k y = \frac{1}{t_k}\int_0^{t_k} T(t)y dt$ for all $y \in E$ and $t_k \to \infty$ when $k \to \infty$. Then,

$$\limsup_{n\to\infty} \langle Fp^*, j(p^* - x_n)\rangle \leq 0. \tag{2.11}$$

Theorem 2.4 *Let E, F, and $\mathcal{F}$ be as in Proposition 2.1. Define a sequence $\{x_n\}$ by* (2.4), *and suppose that conditions* (2.8)-(2.10) *are satisfied. Then, the sequence $\{x_n\}$ converges strongly to the solution p^* of* (1.1).

Remark 2.2 We have improved the result of (2.4) in the sense that we use the mapping $T(t_n)$, instead of using the Bochner integral $T_n x = \frac{1}{t_n}\int_0^{t_n} T(s)xds$. Then, method (2.4) reduces to

$$x_{n+1} = \gamma_n(I - \lambda_n F)x_n + (1-\gamma_n)T(t_n)x_n, \quad n \geq 1, \ x_1 \in E, \ (2.12)$$

where $\lambda_n \in (0,1], \gamma_n \in (0,1)$ and $t_n > 0$ satisfy $\lim_{n\to\infty} t_n = \lim_{n\to\infty} \frac{\gamma_n}{t_n} = 0$. The strong convergence of the method (2.12) was proved under similar conditions on Banach space E, mapping F and nonexpansive semigroups $\{T(s) : s \geq 0\}$ as in Theorem 2.4.

Corolary 2.1 *Assume that the conditions in Theorem 2.2 are satisfied. Consider the sequence $\{x_n\}$ defined by* (2.12), *and suppose that the following conditions are satisfied:*

(i) $\lambda_n \in (0,1], \gamma_n \in (0,1)$ *and* $t_n > 0$;

(ii) $\lim_{n\to\infty} t_n = \lim_{n\to\infty} \frac{\gamma_n}{t_n} = 0$.

Then, $\{x_n\}$ converges strongly to the unique element p^* which solves (1.1).

The iterative method (2.12) is an explicit version of the implicit method (2.2) considered in Theorem 2.2. Next we state and prove a strong convergence theorem for iterative methods (2.5).

Theorem 2.5 *Let E, F, and $\mathcal{F}$ be as in Proposition 2.1. Define the sequence $\{x_n\}$ by* (2.5), *and suppose that conditions* (2.8)-(2.10) *are satisfied. Then, the sequence $\{x_n\}$ converges strongly to the solution p^* of* (1.1).

Remark 2.3

(a) The implicit iterative method has the advantage over the explicit iterative method with mild conditions imposed on parameter sequences but at each iteration we have to solve an equation to find $\{x_k\}$. This difficulty can be overcome by the use of the explicit version (in Section 2.2) of these implicit methods (in Section 2.1) with the same conditions on mappings F, fixed point set $\mathcal{F}$ and Banach space E.

(b) For the sake of completeness, we can cite here some research results with the same approach of constructing solution methods for variational inequalities over the fixed point set of nonexpansive semigroups: Ceng et al. (2008), Chen and He (2007), Yang et al. (2012), Yao et al. (2010). The mathematical framework of the methods mentioned above is a Hilbert space H and a Banach space E with the sequentially weakly continuous normalized duality mapping, respectively. The normalized duality mapping in a Hilbert space H, which is the identity mapping, is certainly sequentially weakly continuous. The normalized duality mapping in a Banach space l^p, $1 < p < \infty$, also has the weakly continuity property. But in general this property does not hold in Banach spaces $L^p[a, b]$, $1 < p < \infty$. Therefore, when considering these methods in a Banach space which does not have a weakly continuous duality mapping j, the convergence of the methods may not be guaranteed. Our results obtained for implicit iterative schemes do not require the weak continuity of the duality mapping of Banach spaces E, and the proof for the convergence of these theorems need to use some different mathematical approaches to overcome the difficulties caused by the geometric characteristics of Banach spaces E and the properties of continuity of the normalized duality mapping j such as the use of the Banach limit μ or sunny nonexpansive retraction Q_C. And thus the scope of applications of the proposed methods can be expanded to $L^p[a, b]$, $1 < p < \infty$ spaces and Sobolev spaces.

(c) Bochner integral of operator $T(s), s \geq 0$ in the form of $\int_0^{t_n} T(s)x_n ds$ can be approximated by Riemann sum (Neerven, 2002).

2.3. Numerical Example

In this section we present a numerical example to illustrate the implicit iterative algorithms (2.1), (2.2) and (2.3), and the explicit hybrid steepest descent methods in the forms of (2.4), (2.5) and (2.12) for variational inequality (1.1). We used 7.0 MATLAB environment software and tested the practical computation on computer DELL INSPIRON, with Intel Core i5, 1.7 GHz CPU and 4GB RAM.

We apply these algorithms studied above for solving the following optimization problem: Find a point$p_* \in C$ such that

$$\varphi(p_*) = \min_{x \in C} \varphi(x), \tag{2.13}$$

Here the function $\varphi : \mathbb{R}^N \to \mathbb{R}$ is assumed to have a strongly monotone and Lipschitz continuous derivative$\bigtriangledown\varphi$ on the Euclidean space $\mathbb{R}^N$, and $C = \mathcal{F}$ is the set of common fixed points of a nonexpansive semigroup $\{T(t),\ t \geq 0\}$ on $\mathbb{R}^N$. As an illustration, we consider the case when $N = 100$, $\varphi(x) = \|x - 1\|^2$ where 1 is the all-ones vector, and $\{T(t),\ t \geq 0\}$ is defined by

$$T(t)x = \begin{pmatrix} \cos(\alpha t) & -\sin(\alpha t) & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ \sin(\alpha t) & \cos(\alpha t) & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & \cos(\alpha t) & -\sin(\alpha t) & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & \sin(\alpha t) & \cos(\alpha t) & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & \cos(\beta t) & -\sin(\beta t) \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & \sin(\beta t) & \cos(\beta t) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ \vdots \\ x_{98} \\ x_{99} \\ x_{100} \end{pmatrix},$$

where $x = (x_1, x_2, \dots, x_{100})^T \in \mathbb{R}^{100}$, and $\alpha, \beta \in \mathbb{R}$ are fixed constants. In this case, $\mathcal{F} = \{x \in \mathbb{R}^{100} :\ x = (0, \dots, 0, x_5, \dots, x_{98}, 0, 0)^T\}$ is a closed and convex subset of $\mathbb{R}^{100}$, and $p_* = (0, 0, 0, 0, 1, \dots, 1, 0, 0)^T$ $\in \mathcal{F} \subset \mathbb{R}^{100}$ is the unique solution of (2.13).

Chapter 3
Regularization Methods for Variational Inequalities over the Set of Common Fixed Points of Nonexpansive Semigroups

In this chapter, we study regularization methods for variational inequality $\mathrm{VI}^*(F, \mathcal{F})$. The contents are presented in four sections. In Section 3.1 and Section 3.2, we propose the Browder–Tikhonov regularization method and the inertial proximal point regularization method for (1.1). In Section 3.3, we construct iterative regularization methods, combining of the Browder–Tikhonov regularization method with the explicit iterative scheme, for variational inequalities over the fixed point set of nonexpansive semigroups. Section 3.4 gives a numerical illustration of the proposed methods. The results of this chapter are taken from articles (3), (4) and (5) in the list of research papers published related to the dissertation.

3.1. Browder–Tikhonov Regularization Method

Banach space settings play such an important role in the past decade of research in the area of regularization theory for inverse and ill-posed problems, and serve as an appropriate framework for such applied problems. The research on regularization methods in Banach spaces was driven by different mathematical viewpoints: on the one hand, there are indeed numerous practical applications where models that use Hilbert spaces, for example by formulating the problem as an operator equation in $L^2[a,b]$-spaces, are not realistic or appropriate. The nature of such applications requires Banach space models working in $L^p[a,b]$-spaces, non-Hilbertian Sobolev spaces, or spaces of continuous functions. In this context, sparse solutions of linear and nonlinear ill-posed operator equations are often to be determined. On the other hand, mathematical tools and techniques typical of Banach spaces can help to overcome the limitations of Hilbert space models.

It is well known that the fixed point problem for nonexpansive mappings is illposed. So the variational inequality problem is, in general, ill-posed too. To solve the class of these problems, we have to use stable methods, as the Tikhonov regularization method. In 2012, Buong and Phuong (2012) studied an implicit and an explicit regularization method for solving a variational inequality problem defined in a real reflexive and strictly convex Banach space E. In these methods, the feasible set is defined as the common fixed points associated with a family of nonexpansive mappings. These regularization methods are based on a V-mapping and constructed as a simple iteration combined with a Browder–Tikhonov regularization. Recently, Thuy (2015) has improved Buong and Phuongs results by considering an implicit and an explicit scheme based on a S-mapping which is simpler to compute than the V-mapping. In this work, our aim is to study an extension of Buong and Phuongs results as well as Thuys results for solving the variational inequality problem whose constraint set is given as the common fixed points of a nonexpansive semigroup defined on a Banach space.

Method 3.1. *The regularized equation for problem* (1.1) *is given as follows:*

$$A_n x_n + \varepsilon_n F x_n = 0, \;\; n \geq 0 \tag{3.1}$$

where $A_n = I - T_n$, *and* T_n *is defined by*

$$T_n x = \frac{1}{t_n} \int_0^{t_n} T(s) x ds \quad \textit{for all } x \in E, \tag{3.2}$$

with $\{t_n\}$, $\{\varepsilon_n\}$ *are sequences of positive real numbers satisfying* $t_n \to \infty$ *and* $\varepsilon_n \to 0$ *as* $n \to \infty$.

Theorem 3.1 *Let* $F : E \to E$ *be an* η*-strongly accretive and* γ*-strictly pseudocontractive mapping with* $\eta + \gamma > 1$. *Let* $\{T(s) : s \geq 0\}$ *be a nonexpansive semigroup on* E *such that* $\mathcal{F} = \cap_{s \geq 0} \mathrm{Fix}(T(s))$ *is nonempty. Then,* *(i) for each $t_n > 0$ and each ε_n, regularized equation (3.1) has a unique solution x_n.*

(ii) if sequences $\{t_n\}$ and $\{\varepsilon_n\}$ are chosen such that

$$\lim_{n\to\infty} t_n = +\infty \quad \text{and} \quad \lim_{n\to\infty} \varepsilon_n = 0,$$

then $\{x_n\}$ converges strongly to the element $p_* \in \mathcal{F}$ which solves (1.1).

(iii) Furthermore, we have the following estimation

$$\|x_n - x_m\| \leq \left(\frac{|\varepsilon_m - \varepsilon_n|}{\varepsilon_n} + 2\frac{|t_m - t_n|}{\varepsilon_n t_m}\right)\frac{M_1}{\eta}. \tag{3.3}$$

where M_1 is a positive constant, x_n and x_m are regularized solutions of (3.1) associated to parameters t_n, ε_n and t_m, ε_m, respectively.

3.2. The Inertial Proximal Point Regularization Method

The inertial proximal point method was proposed by Alvarez (2000) for the convex optimization problem in Hilbert spaces. After that, Attouch and Alvarez (2001) used this scheme to consider the zero point problem of maximal monotone A in H in the form

$$0 \in c_n A z_{n+1} + z_{n+1} - z_n - \gamma_n(z_n - z_{n-1}), \quad z_0, z_1 \in H.$$

When $\gamma_n = 0$, the method reduces to the proximal point method studied by Rockafellar in 1976 for the stationary problem of a maximal monotone operator A.

Based on the Browder–Tikhonov regularization method (3.1), we combine it with the inertial proximal point method to generate an equation of $\{z_n\}$ as follows.

Method 3.2. *Start from initial guesses $z_0, z_1 \in E$ arbitrarily, we construct an iterative equation of $\{z_n\}$ as follows:*

$$c_n(A_n + \varepsilon_n F)(z_{n+1}) + z_{n+1} - z_n = \gamma_n(z_n - z_{n-1}), \tag{3.4}$$

where $\{c_n\}$ and $\{\gamma_n\}$ are positive parameters satisfying some appropriate conditions.

3.2.1. The Strong Convergence

Theorem 3.2 *Let E, $\mathcal{F}$, and A be given as in Theorem 3.1. Assume that the parameters c_n, ε_n, t_n and γ_n are chosen such that*

(*i*) $0 < m < c_n < M,\ 0 \leq \gamma_n < \gamma_0;\ 1 \geq \varepsilon_n \searrow 0,\ t_n \to \infty;$

(*ii*) $\sum_{n=1}^{\infty} b_n = +\infty,\ b_n = \eta c_n \varepsilon_n/(1+\eta c_n \varepsilon_n);$

(*iii*) $\lim_{n\to\infty} \gamma_n b_n^{-1}\|z_n - z_{n-1}\| = 0;$

(*iv*) $\lim_{n\to\infty} \frac{\varepsilon_n - \varepsilon_{n+1}}{\varepsilon_n^2} = \lim_{n\to\infty} \frac{|t_n - t_{n+1}|}{\varepsilon_n^2 t_{n+1}} = 0.$

Then the sequence $\{z_n\}$ defined by (3.4) *converges strongly to p^* as $n \to +\infty$, which solves variational inequality* (1.1)*.*

Remark 3.1

(*a*) The sequences $\{\varepsilon_n\}$ and $\{\gamma_n\}$ are defined by

$$\varepsilon_n = (1+n)^{-p},\ 0 < p < 1/2,\ \gamma_n = (1+n)^{-\tau}\frac{\|z_n - z_{n-1}\|}{1+\|z_n - z_{n-1}\|^2}$$

with $\tau > 1+p$ satisfy all conditions of Theorem 3.2 (see Buong (2008) for more details).

(*b*) In the case when $\{T(s) : s \geq 0\}$ is a nonexpansive semigroup over a closed and convex subset C in E, in [2], we considered the following regularized equation:

$$(I - T_n Q_C)x_n + \varepsilon_n F x_n = 0. \tag{3.5}$$

With the same conditions as stated in Theorem 3.1, we also obtained results similar to (*i*), (*ii*) and (*iii*) of Theorem 3.1.

(*c*) In the case when $\mathcal{F} = \cap_{i=1}^{\infty}\mathrm{Fix}(T_i)$, the set of common fixed points of countably infinite family of nonexpansive mappings $(T_i)_{i=1}^{\infty}$, by using V-mapping, Buong and Phuong (2012) considered the regularized equation for (1.1) as follows:

$$(I - V_n)x_n + \varepsilon_n F x_n = 0. \tag{3.6}$$

After that, Thuy (2015) improved Buong–Phuong's results for the similar problem by using S-mapping instead of V-mapping.

When $E \equiv H$, we studied regularization methods for finding a x_*-minimal norm common fixed point of nonexpansive semigroup $\{T(s) : s \geq 0\}$ on a closed and convex subset C in Hilbert space H with $\mathcal{F} = \cap_{s\geq 0}\mathrm{Fix}(T(s)) \neq \emptyset$ without using the Bochner integral T_n. The problem is stated as follows: *Find a point $p \in \mathcal{F}$ satisfying*

$$\|x_* - p\| = \min_{y\in\mathcal{F}} \|x_* - y\|, \tag{3.7}$$

where x_* is an element in H but not in $\mathcal{F}$.

Inspired from the idea of regularizing variational inequalities over the fixed point set of a nonexpansive semigroup $\{T(s) : s \geq 0\}$, we construct the regularized equation for problem (3.7) without using the Bochner integral $T_n x = \frac{1}{t_n}\int_0^{t_n} T(s)x ds$ under the following form: *Find elements $x_n \in H$ such that*

$$A^C(t_n)x_n + \varepsilon_n(x_n - x_*) = 0, \quad A^C(t_n) = I - T(t_n)P_C, \tag{3.8}$$

where I is the identity mapping of H, P_C is the metric projection from H onto C, and $\{t_n\}, \{\varepsilon_n\}$ are sequences of positive real numbers satisfying some appropriate conditions.

Theorem 3.3 [4] *Let C be a nonempty closed and convex subset of a real Hilbert space H and let $\{T(t) : t \geq 0\}$ be a nonexpansive semigroup on C such that $\mathcal{F} = \cap_{t\geq 0} \mathrm{Fix}(T(t)) \neq \emptyset$. Then we have the following statements:*
(i) For each ε_n, $t_n > 0$, problem (3.8) *has a unique solution x_n.*
(ii) If t_n and ε_n are chosen such that

$$\liminf_{n\to\infty} t_n = 0,\ \limsup_{n\to\infty} t_n > 0,\ \lim_{n\to\infty}(t_{n+1} - t_n) = 0,\ \textit{and}\ \lim_{n\to\infty}\varepsilon_n = 0,$$

then the sequence $\{x_n\}$ converges strongly, as $n \to +\infty$, to p, the unique solution of (3.7).

Furthermore, we have the following evaluation for $\|x_n - x_m\|$ with two regularized solutions x_n and x_m of (3.7) as stated in Lemma 3.1. This result is used to prove the convergence of the proximal point regularization method and the iterative regularization that will be considered in Theorem 3.4 and Theorem 3.6.

Lemma 3.1 *Let H, $C, \{T(s) : s \geq 0\}$ and $\mathcal{F}$ be defined as in Theorem 3.3. Let x_n and x_m be two regularized solutions of equation* (3.7). *If $\|T(t)x - T(h)x\| \leq |t-h|\gamma(x)$ for each $x \in C$, where $\gamma(x)$ is a bounded function, then*

$$\|x_n - x_m\| \leq \frac{|\varepsilon_n - \varepsilon_m|}{\varepsilon_n}\|y - x_*\| + \frac{|t_n - t_m|}{\varepsilon_n}\gamma_1$$

for each ε_n, ε_m, t_n, $t_m > 0$, $y \in \mathcal{F}$, and some positive constant γ_1.

The second scheme is a combination of the studied regularization method with the proximal point scheme proposed by Rockafellar (1976), called the regularization proximal point algorithm. The idea used in this paper is to generate an approximation sequence for problem (3.7) as follows. For any given point $z_0 \in H$, the sequence $\{z_n\}$ is defined by:

$$c_n[A^C(t_n)z_{n+1} + \varepsilon_n(z_{n+1} - x_*)] + z_{n+1} = z_n, \quad n \geq 0, \tag{3.9}$$

where $\{c_n\}$ is a bounded sequence of real positive numbers.

Theorem 3.4 *Let C be a nonempty closed convex subset of real Hilbert space H and let $\{T(t) : t \geq 0\}$ be a nonexpansive semigroup on C such that $\mathcal{F} = \cap_{t\geq 0}\mathrm{Fix}(T(t)) \neq \emptyset$. Assume that the parameters c_n, t_n and ε_n are chosen such that*
(i) $0 < m < c_n < M$;
(ii) $\liminf\limits_{n\to\infty} t_n = 0$, $\limsup\limits_{n\to\infty} t_n > 0$, $\lim\limits_{n\to\infty}(t_{n+1} - t_n) = 0$;
(iii) $1 \geq \varepsilon_n\ \forall n$, $\sum_{n=0}^{\infty} \varepsilon_n = +\infty$, *with* $\lim\limits_{n\to\infty} \frac{|\varepsilon_n - \varepsilon_{n+1}|}{\varepsilon_n^2} = \lim\limits_{n\to\infty} \frac{|t_n - t_{n+1}|}{\varepsilon_n^2} = 0$; *and*
$\|T(t)x - T(h)x\| \leq |t-h|\gamma(x)$ *for each $x \in C$, where $\gamma(x)$ is a bounded functional.*

Then, the sequence $\{z_n\}$ defined by (3.9) *converges strongly, as $n \to +\infty$, to the element $p \in \mathcal{F}$ which solves* (3.7).

When $C \equiv H$ then (3.8) and (3.9) reduce to the following methods:

$$(I - T(t_n))x_n + \varepsilon_n(x_n - x_*) = 0,$$
$$c_n[(I - T(t_n))z_{n+1} + \varepsilon_n(z_{n+1} - x_*)] + z_{n+1} = z_n, \quad n \geq 0.$$

3.3. Iterative Regularization Method

In the third method, we proposed an explicit iterative scheme based on the regularization method (3.1). Start with a given point $w_1 \in E$ and define a sequence w_n iteratively by

$$w_{n+1} = w_n - \beta_n[A_n w_n + \varepsilon_n F w_n], \quad n \geq 1, \tag{3.10}$$

where $A_n = I - T_n$, and the sequence $\{\beta_n\}$ satisfies some control conditions.

Theorem 3.5 *Let E be a uniformly convex and q-uniformly smooth Banach space for a fixed q with $1 < q \leq 2$, and let $\mathcal{F}$ and F be as in Theorem 3.1. Assume that*

$$(i)\ 0 < \beta_n < \beta_0,\ \varepsilon_n \searrow 0, \quad \lim_{n\to\infty} \frac{\varepsilon_n - \varepsilon_{n+1}}{\varepsilon_n^2 \beta_n} = \lim_{n\to\infty} \frac{|t_n - t_{n+1}|}{\beta_n \varepsilon_n^2 t_n} = 0,$$

$$(ii)\ \sum_{n=0}^{\infty} \varepsilon_n \beta_n = \infty, \quad \limsup_{n\to\infty} C_q \beta_n^{q-1} \frac{(2 + \varepsilon_n L)^p}{\varepsilon_n \eta} < 1,$$

where C_q is the uniformly smooth constant of E. Then, the sequence $\{w_n\}$ defined by (3.10) *converges strongly, as $n \to +\infty$, to p^*, the solution of* (1.1).

Remark 3.2

(a) The sequences $\varepsilon_n = (1 + n)^{-p},\ 0 < 2p < 1$ and $\beta_n = \gamma_0 \varepsilon_n$ with

$$0 < \gamma_0 < \frac{1}{C_q^{1/q-1}(2 + \varepsilon_0)^{q/q-1}}$$

satisfy all conditions of Theorem 3.5 when $q = 2$. In the case $1 < q < 2$, $\varepsilon_n = (1 + n)^{-p}$ where $p < (q - 1)/2q$ and $\beta_n = \gamma_0 \varepsilon_n^{1/q-1}$ also satisfy all conditions of the theorem (see Buong and Phuong (2012) for more details).

(b) Authors Buong and Phuong (2012) also used V-mapping to generate an iterative regularization method for approximated solution of (1.1), while Thuy used S-mapping for the same problem over the feasible set $C := \mathcal{F} = \cap_{i=1}^{\infty} \mathrm{Fix}(T_i)$, the fixed point set of a countably infinite family of nonexpansive mappings.

(c) Based on the idea of combining the Browder–Tikhonov regularization method with an iterative scheme to establish iterative regularization method for common fixed point of a nonexpansive semigroup in Hilbert spaces in the form of problem (3.7) in Hilbert spaces, we introduce the following iterative sequence: *Starting from a given point $w_0 \in H$, a sequence $\{w_n\}$ is generated iteratively by the following rule:*

$$w_{n+1} = w_n - \beta_n[A^C(t_n)w_n + \varepsilon_n(w_n - x_*)], \ n \geq 0, \ w_0 \in H, \quad (3.11)$$

where $\{\beta_n\}$ is a sequence of positive real numbers satisfying some control condition.

Theorem 3.6 *Let C be a nonempty closed convex subset of a real Hilbert space H and let $\{T(t) : \ t \geq 0\}$ be a nonexpansive semigroup on C such that $\mathcal{F} = \bigcap_{t\geq 0} \mathrm{Fix}(T(t)) \neq \emptyset$. Assume that the following conditions hold:*
(i) $\beta_n \leq \frac{\varepsilon_n}{4+4\varepsilon_n+4\varepsilon_n^2}$ *for all n,* $\lim\limits_{n\to\infty} \frac{|\varepsilon_n-\varepsilon_{n+1}|}{\varepsilon_n^2\beta_n} = \lim\limits_{n\to\infty} \frac{|t_n-t_{n+1}|}{\varepsilon_n^2\beta_n} = 0$, *and* $\sum_{n=0}^{\infty} \varepsilon_n\beta_n = +\infty, \quad \varepsilon_n \to 0;$
(ii) $\liminf\limits_{n\to\infty} t_n = 0, \ \limsup\limits_{n\to\infty} t_n > 0, \ \lim\limits_{n\to\infty} (t_{n+1} - t_n) = 0;$
(iii) $\|T(t)x - T(h)x\| \leq |t-h|\gamma(x)$ *for each $x \in C$, where $\gamma(x)$ is a bounded functional.*
Then, the sequence $\{w_n\}$ defined by (3.11) *converges strongly, as $n \to +\infty$, to the unique element $p \in \mathcal{F}$ which solves* (3.7).

If $C \equiv H$, then (3.11) becomes

$$w_{n+1} = w_n - \beta_n[(I - T(t_n))w_n + \varepsilon_n(w_n - x_*)], \ n \geq 0, \ w_0 \in H.$$

3.4. Numerical Example

In this section, we use regularization methods (3.1), (3.4) and (3.10) to solve variational inequalities (1.1) and regularization methods (3.8), (3.9) and (3.11) to find a common fixed point of a nonexpansive semigroup considered in problem (3.7) of Chapter 2.

CONCLUSION AND RECOMMENDATION

Conclusion: The dissertation studies the problem of solving variational inequalities over the set of common fixed points of nonexpansive semigroups in Banach spaces by the hybrid steepest descent method and regularization methods in Banach spaces without condition of sequentially weakly continuous property of normalized duality mappings.

1. For the hybrid steepest descent method: We propose three implicit and two explicit hybrid steepest descent methods which strongly converge to the unique solution of an accretive variational inequality over the feasible set of common fixed points of nonexpansive semigroups on Banach spaces with uniformly Gteaux differentiable norm. The results presented in this work improve some known results of Buong and Quynh Anh (2011), Buong and Thuy Duong (2011) in Hilbert spaces; Chen and He (2007), Ceng-Ansari-Yao (2008) in Banach spaces.

2. For regularization methods: We present and prove strong convergence theorems for the Browder-Tikhonov regularization method for accretive variational inequalities over the set of common fixed points of nonexpansive semigroups on Banach spaces with uniformly Gteaux differentiable norm; combine the Browder-Tikhonov regularization method with the inertial proximal point method for the same problems. A coupling method of an explicit iterative scheme and the Browder-Tikhonov regularization method is proposed to solve accretive variational inequalities in q-uniformly smooth Banach spaces.

3. Give and discuss numerical examples for the proposed methods.

Recommendation

1. Weaken the assumptions on the variational inequality problem mapping.

2. Study stopping criteria of the proposed methods and compare the convergence rate of these methods.

3. Extend the proposed methods to split variational inequalities.

RESEARCH PAPERS PUBLISHED RELATED TO THE DISSERTATION

(1) Nguyen Buong, Nguyen Thi Thu Thuy and Pham Thanh Hieu (2013), "An explicit iteration method for a class of variational inequalites in Banach spaces", *Proceedings of the 15th National Conference on Some Selected Problems on Information Technology and Communication: Scientific Computing*, pp. 6-10, Science and Engineering Publishing House, Hanoi.

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(3) Pham Thanh Hieu (2014), "A regularization method for variational inequalities in Banach spaces", *Journal of Science and Technology, Thai Nguyen University*, Vol. 126(12), pp. 87-92.

(4) Pham Thanh Hieu and Nguyen Thi Thu Thuy (2015), "Regularization methods for nonexpansive semigroups in Hilbert spaces", *Vietnam J. Math.*, DOI 10.1007/s10013-015-0178-3 (SCOPUS), Published online: 18 December 2015.

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