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LIST OF PUBLICATION RELATED TO PhD DISSERTATION

1. Nguyen Thi Quynh Anh (2009), "Quasi optimization problem of type I and quasi optimization problem of type II", *Tạp chí Khoa Công nghệ Đại học Thái Nguyên*, 56 (8), 45-50.
2. Nguyen Buong and Nguyen Thi Quynh Anh (2011), "An implicit iteration method for variational inequalities over the set of common fixed points for a finite family of nonexpansive mappings in Hilbert spaces", *Hindawi Publish Coporation, Fixed point thoery applications*, volume 2011, article ID 276859.
3. Nguyen Xuan Tan and Nguyen Thi Quynh Anh (2011), "Generalized quasi-equilibrium problems of type 2 and their applications", *VietNam journal of mathematics*, volume 39, 1-25.
4. Nguyen Thi Quynh Anh and Nguyen Xuan Tan (2013), "On the existence of solutions to mixed Pareto quasivariational inclusion problems", *Advances in Nonlinear variational Inequalities*, volume 16, Number 2, 1-22.
5. Nguyen Thi Quynh Anh (2014), "Modified viscosity approximation methods with weak contraction mapping for an infinite family of nonexpansive mappings", *East - West journal of mathematics*, volume 16, No 1, 1-13.

INTRODUCTION

Vector optimization theory is formed from ideas about economic equilibrium (1881) and value theory (1909) of Edgeworth. But since the 1950s onwards, after the works by Kuhn - Tucker 1951 of the equilibrium value and Pareto optimization by Debreu (1954), vector optimization theory has been really welcomed as a new branch of modern mathematics and has multiple applications in practice.

Let D be a nonemptyset in space X, $f : D \to \mathbb{R}$ be a real function. The following minimum problems of function f in D could be seen as the central problem in the theory of optimization: Find $\bar{x} \in D$ such that

$$f(\bar{x}) \leq f(x) \text{ for all } x \in D. \tag{0.1}$$

Relating to this problem, we have known variational inequalities which were initially studied by Stampacchia in 1980: Let $D \subseteq \mathbb{R}^n$, $G : D \to \mathbb{R}^n$ is a single-valued mappings. The problem is as follows: Find $\bar{x} \in D$ such that

$$\langle G(\overline{x}), x - \overline{x} \rangle \geq 0 \text{ for all } x \in D. \tag{0.2}$$

Let $T : D \to X$ be a single-valued mapping. The fixed point problem is formed: Find $\bar{x} \in D$ such that

$$\bar{x} = T(\bar{x}). \tag{0.4}$$

If T is a continuous mappings and $G := I - T$, where $I : D \to D$ denotes the identity mapping, then the fixed point problem (0.4) is equivalent to variational inequality problem (0.2).

In 1994, Blum E. and Oettli W. introduced equilibrium problem and showed sufficient conditions on the existence of its solutions: Let X be a real topological locally convex Hausdorff, $D \subseteq X, \varphi : D \times D \to \mathbb{R}$. Find $\bar{x} \in D$ such that

$$\varphi(t, \bar{x}) \geq 0 \text{ for all } t \in D. \tag{0.5}$$

The typical instances of this problem are fixed point problem, variational inequalities, Nash equilibrium problem,...

In 2002, Nguyen Xuan Tan and Guerraggio A. introduced quasi-optimization problem and showed sufficient conditions on the existence of its solutions: Let X, Z be topological locally convex Hausdorff, $D \subseteq X, K \subseteq Z$ be nonempty subsets, $S : D \times K \to 2^D, T : D \times K \to 2^K$ be multivalued mappings, $F : K \times D \times D \to \mathbb{R}$ be a function. Find $(\bar{x}, \bar{y}) \in D \times K$ such that

$$\begin{aligned} &1) \quad \bar{x} \in S(\bar{x}, \bar{y}),\ \bar{y} \in T(\bar{x}, \bar{y}), \\ &2)\ F(\bar{y}, \bar{x}, \bar{x}) = \min_{t \in S(\overline{x}, \overline{y})} F(\bar{y}, \bar{x}, t). \end{aligned} \tag{0.6}$$

Problem (0.6) is more generalized than (0.5). In case F is independent in y, $F(x, x) = 0$ for all $x \in D$, we are setting $S(x, y) \equiv D$ and $\varphi(t, x) = F(x, t)$ for all $x, t \in D$. From the fact that (0.6) implies $0 = F(\bar{x}, \bar{x}) \leq F(\bar{x}, t)$ for all $t \in D$, that means $\varphi(t, \bar{x}) \geq 0$ for all $t \in D$, (0.5) is satisfied.

Problem (0.1) has been extended for vectors: Let X, Y be a real topological locally convex Hausdorff spaces, $D \subseteq X$, $C \subseteq Y$ be a cone. We consider patiel order relation in Y is generated by cone C: $x \succeq y$ iff $x - y \in C$. We define the set of α effective points in $A \subseteq Y$, denoted by $\alpha Min(A/C)$, called α effective points set of the set A to C, (α is ideal, proper, Pareto and weak). The problem: Find $\bar{x} \in D$ such that

$$F(\bar{x}) \in \alpha \mathrm{Min}(F(D)/C), \tag{0.7}$$

with $F : D \to Y$, is called quasi- optimization α vector problems. $\bar{x}$ and $F(\bar{x})$ are called optimal solution and optimal value α of (0.7), respectively.

In 1985, Nguyen Xuan Tan extended the problem (0.2) for valued mappings and constraints domain D dependent in S: Let $D \subseteq X$ be a subset of vector topological convex locally Hausdorff space X with duality space X^*, $S : D \to 2^D, P : D \to 2^{X^*}$ be multivalued mappings and $\varphi : D \to \mathbb{R}$ be a function. The problem: Find $\bar{x} \in D, \bar{x} \in S(\bar{x})$ and $\bar{y} \in P(\bar{x})$ such that

$$\langle \overline{y}, x - \overline{x} \rangle + \varphi(x) - \varphi(\overline{x}) \geq 0 \text{ for all } x \in S(\overline{x}), \tag{0.8}$$

is called quasivariational multivalued inequality.

In 1998, Nguyen Xuan Tan và Phan Nhat Tinh extended the problem (0.3) for vectors. Next, in 2000, Nguyen Xuan Tan and Nguyen Ba Minh extended for multivalued mappings and they proved a theorem on the existence of solutions to Blum-Oettli problem.

In 2007, Lin J. L. and Nguyen Xuan Tan stated quasivariational inclusion problems of type 1. In 2004, Dinh The Luc and Nguyen Xuan Tan stated quasivariational inclusion problems of type 2. Then, Bui The Hung and Nguyen

Xuan Tan proved Theorems on the existence of solutions to Pareto quasivariational inclusion problems of type 1 and type 2 (2012). These results implie many results for related problems.

Following Truong Thi Thuy Duong and Nguyen Xuan Tan's studies on generalized quasi-equilibrium problem of type 1, in 2011, we stated generalized quasi-equilibrium problem of type 2:
Find $\bar{x} \in D$ such that $\bar{x} \in P_1(\bar{x})$ and

$$0 \in F(y, \bar{x}, t) \text{ for all } t \in P_2(\bar{x}) \text{ and } y \in Q(\bar{x}, t).$$

The above problems contain quasivariational inclusion, quasi-equilibrium and quasivariational relation problems of type 1 and type 2 like specific cases.

Truong Thi Thuy Duong's dissertation obtained the existence of solutions to mixed generalized quasi-equilibrium problem: Find $(\bar{x}, \bar{y}) \in D \times K$ such that

$$\begin{aligned}
&1) \quad \bar{x} \in S(\bar{x}, \bar{y}), \bar{y} \in T(\bar{x}, \bar{y}),\\
&2) \quad 0 \in F(\bar{y}, \bar{y}, \bar{x}, t) \text{ for all } t \in S(\bar{x}, \bar{y}),\\
&3) \quad 0 \in G(y, \bar{x}, t) \text{ for all } t \in P(\bar{x}), y \in Q(\bar{x}, t),
\end{aligned}$$

where X, Y_1, Y_2, Z be vector topological convex locally Hausdorff spaces, $F : K \times K \times D \times D \to 2^Y, G : K \times D \times D \to 2^Y$ and P, Q, S, T be the same as above mappings. Truong Thi Thuy Duong gives strictly hypotheses (such as hypothese *iv)* in Theorem 4.2.2). The objectves of dissertation were to state and prove the existence of solutions to generalized quasi-equilibrium problem of type 2, find relations to other multivalued optimal problems, study mixed Pareto quasivariational inclusion problems with hypotheses easy to test and find new implicit iteration methods for finding a solution to variational inequality problems.

Chapter 1 introduces some basic knowledge on multivalued analysis which used in Dissertation's main chapter.

Chapter 2 is for generalized quasi-equilibrium problem: generalized quasi-equilibrium problem of type 2 (Theorem 2.3.1), quasivariational relation problem (Corolary 2.4.1), undirected quasi-equilibrium problem (Corollary 2.4.2), ideal quasivariational inclusions (Corollaries 2.4.3 and 2.4.4), ideal quasi-equilibrium problems (Corollaries 2.4.5 and 2.4.6). In special case, we show some results on the existence of solutions to upper (lower) Pareto (weak) quasi-equilibrium problems of type 1 and type 2 related to monotone mappings (see Theorems 2.4.2, 2.4.3, 2.4.4, 2.4.5).

Chapter 3 shows the existence of solutions to mixed Pareto quasivariational inclusion problems (Theorems 3.2.1, 3.2.2, 3.2.3 and 3.2.4) and the existence of solutions to related problems, such as: systems of Pareto quasi-variational inclusion problems, Pareto quasi optimization problems, mixed Pareto quasi-equilibrium problems.

In Chapter 4, we present some implicit iteration methods to find solutions of variational inequalities (see Theorems 4.2.1, 4.2.2, 4.2.3).

Chapter 1. SOME BASIC KNOWLEDGE

Chapter 1 shows real topological locally convex Hausdorff spaces, some definitions, some properties of cones and set-valued mappings.

Chapter 2. GENERALIZED QUASI-EQUILIBRIUM PROBLEMS

In this chapter, Section 2.1, we introduce generalized quasi-equilibrium problems related to multivalued mappings. In Ssection 2.2, we consider the existence of solutions to these problems. Sections 2.2 and 2.4 show that vector multivalued optimization problems, variational inclusion problems, quasi-equilibrium problems of type 1 and type 2,... are quasi-equilibrium problems. Section 2.5 obtains some results of the stability of the solutions to generalized quasi-equilibrium problems which are dependent on parameters.

2.1. Introduction to problems

Throughout this chapter, X, Z and Y are supposed to be real topological locally convex Hausdorff spaces, $D \subset X, K \subset Z$ are nonempty subsets. Given multivaled mappings $S : D \times K \to 2^D, T : D \times K \to 2^K; P_1 : D \to 2^D, P_2 : D \to 2^D, Q : K \times D \to 2^K$ and $F_1 : K \times D \times D \times D \to 2^Y, F : K \times D \times D \to 2^Y$, we are interested in the following problems:

1/ Find $(\bar{x}, \bar{y}) \in D \times K$ such that

i) $\bar{x} \in S(\bar{x}, \bar{y})$,

ii) $\bar{y} \in T(\bar{x}, \bar{y})$,

iii) $0 \in F_1(\bar{y}, \bar{x}, \bar{x}, z)$, for all $z \in S(\bar{x}, \bar{y})$.

This problem is called a generalized quasi-equilibrium problem of type 1.

2/ Find $\bar{x} \in D$ such that

1) $\bar{x} \in P_1(\bar{x})$,

2) $0 \in F(y, \bar{x}, t)$, for all $t \in P_2(\bar{x})$ and $y \in Q(\bar{x}, t)$.

This problem is called a generalized quasi-equilibrium problem of type 2.

3/ Find $(\bar{x}, \bar{y}) \in D \times K$ such that

1) $\bar{x} \in S(\bar{x}, \bar{y})$, $\bar{y} \in T(\bar{x}, \bar{y})$,

2) $0 \in F_1(\bar{y}, \bar{x}, \bar{x}, z)$ for all $z \in S(\bar{x}, \bar{y})$,

3) $0 \in F(y, \bar{x}, t)$ for all $t \in P_2(\bar{x})$ and $y \in Q(\bar{x}, t)$.

In the above problems, the multivalued mappings S, T, P_1, P_2 and Q are constraints, F_1 and F are utility multivalued mappings that are often determined by equalities and inequalities, or by inclusions, not inclusions and intersections of other multivalued mappings, or by some relations in product spaces. The generalized quasi-equilibrium problems of type 1 are studied by Truong Thi Thuy Duong (2011). In this chapter, we consider the existence to solutions of the second ones. The typical examples of generalized quasi-equilibrium problems of type 2 are the following:

2.2. The problems related to generalized quasi-equilibrium problems

This section shows typical examples of generalized quasi-equilibrium problems of type 2, such as: undirected quasi-equilibrium problem, Minty quasivariational problem, ideal quasivariational inclusion problems, ideal quasi-equilibrium problems, quasivariational relation problem, differential inclusion, optimal control, Nash quasi-equilibrium problem in noncoorperation games,...

2.3. The sufficient conditions on the existence of solutions to generalized quasi-equilibrium problems type 2

In this section, we apply Theorem Fan-Browder to prove the existence of the solutions to generalized quasi-equilibrium problems type 2, there by deduces some results to the relatedproblems.

Theorem 2.3.1. *The following conditions are sufficient for* $(GEP)_{II}$ *to have a solution:*

i) D is a nonempty convex compact subset;

ii) $P_1 : D \to 2^D$ is a multivalued mapping with a nonempty closed fixed point set $D_0 = \{x \in D | \; x \in P_1(x)\}$ in D;

iii) $P_2 : D \to 2^D$ is a multivalued mapping with $P_2^{-1}(x)$ open and the convex hull $coP_2(x)$ of $P_2(x)$ is contained in $P_1(x)$ for each $x \in D$;

iv) *For any fixed* $t \in D$, *the set*

$$B = \{x \in D|\ 0 \notin F(y,x,t),\ \textit{for some}\ \ y \in Q(x,t)\}$$

is open in D;

v) $F : K \times D \times D \to 2^Y$ *is a* $Q - KKM$ *multivalued mapping.*

Theorem 2.3.2 shows that if we replace the opennees of $P_2^{-1}(x)$ with the lower semicontinuity of P_2, generalized quasi-equilibrium problems of type 2 have solutions.

2.4. The sufficient conditions on the existence of solutions to interest problems

Several applications of the above theorem in the solution existence of quasi-equilibrium, variational inclusion problems,... can be shown in the following corollaries.

2.4.1. The quasi-variational relation problem

Corollary 2.4.1 introduces another proof of Đinh The Luc's result (2008).
Corollary 2.4.1.*Let* D, K, P_1, P_2 *be as Theorem 2.3.1,* $Q(.,t)$ *be an upper semicontinuous mapping for any* $t \in D$, $\mathcal{R}$ *be a relation linking* $y \in K, x \in D$ *and* $t \in D$. *In addition, assume:*

i) *For any fixed* $t \in D$ *the relation* $R(.,.,t)$ *linking elements* $y \in K, x \in D$ *is closed;*

ii) $\mathcal{R}$ *is Q-KKM.*

Then, there exists $\bar{x} \in D$ *such that* $\bar{x} \in P_1(\bar{x})$ *and*

$$\mathcal{R}(y,\bar{x},t)\ \textit{holds for all}\ \ t \in P_2(\bar{x})\ \ \textit{and}\ \ y \in Q(\bar{x},t).$$

2.4.2. Undirected quasi-equilibrium problems The below result was

directly proved by the Theorem 2.3.1 and it was also Nguyen Xuan Tan and Dinh The Luc's results published in 2004.

Corollary 2.4.2. *Let D, K, P_1, P_2 be as in Theorem 2.3.1, $Q(.,t)$ be a lower semicontinuous mapping for any $t \in D$. Let $\Phi : K \times D \times D \to R$ be a real diagonally $(Q, R_+)-$ quasiconvex-like in the third variable function with $\Phi(y,x,x) = 0$, for all $y \in K, x \in D$. In addition, assume that for any fixed $t \in D$ the function $\Phi(.,.,t) : K \times D \to R$ is upper semicontinuous. Then, there exists $\bar{x} \in D$ such that $\bar{x} \in P_1(\bar{x})$ and*

$$\Phi(y, \bar{x}, t) \geq 0 \text{ for all } t \in P_2(\bar{x}) \text{ and } y \in Q(\bar{x}, t).$$

In the next corollaries in the Sections 2.4.3 and 2.4.4, we suppose that C be a closed convex cone in Y.

2.4.3. Ideal quasi-variational inclusions

Theorem 2.3.1 gives some results on the existence of the solutions to upper (lower) ideal quasivariational inclusions. This result implies Dinh The Luc and Nguyen Xuan Tan's results published in 2004.

Corollary 2.4.3.*Let D, K, P_1, P_2 be as in Theorem 2.3.1 and $Q : D \times D \to 2^K$ be such that for any fixed $t \in D$, the multivalued mapping $Q(.,t) : D \to 2^K$ be lower semicontinuous. Let $G, H : K \times D \times D \to 2^Y$ be multivalued mappings with compact values and $G(y,x,x) \subseteq H(y,x,x)+C$, for any $(y,x) \in K \times D$. In addition, assume:*

- *i) For any fixed $t \in D$, the multivalued mapping $G(.,.,t) : K \times D \to 2^Y$ is lower $(-\mathcal{C})-$continuous and the multivalued mapping $N : K \times D \to 2^Y$, defined by $N(y,x) = H(y,x,x)$, is upper $C-$ continuous;*
- *ii) G is diagonally upper (Q, C)-quasiconvex-like in the third variable.*

Then, there exists $\bar{x} \in D$ such that $\bar{x} \in P_1(\bar{x})$ and

$$G(y, \bar{x}, t) \subseteq H(y, \bar{x}, \bar{x}) + C, \quad \text{for all } t \in P_2(\bar{x}) \text{ and } y \in Q(\bar{x}, t).$$

Similarly, we obtain results for lower quasivariational inclusions. Section 2.4.4 shows the results on the existence the solutions of ideal quasi-equilibrium problems.

2.4.5. Pareto and weakly quasi-equilibrium problems

This section shows the existence of soluions to Pareto (weak) quasi-equilibrium problems (in both cases: the utility multivalued mappings be $\mathcal{C}$-convex and $\mathcal{C}$–quasiconvex like). We need the following lemmas in the sequel.

Lemma 2.4.1. *Let $F : K \times D \times D \to 2^Y$ be a multivalued mappings with nonempty valued, $\mathcal{C} : K \times D \to 2^Y$ be a cone multivalued mappings with $F(y, x, x) \subseteq (\mathcal{C}(y, x))$ for any $x \in D, y \in K$. In addition, assume that:*

i) For any fixed $x \in D, y \in K, F(y, ., x) : D \to 2^Y$ is upper $\mathcal{C}(y, .)$-hemicontinuity;

ii) For any fixed $y \in K$, $F(y, ., .)$ is lower $\mathcal{C}(y, .)$- strong pseudomonotone;

iii) For any fixed $y \in K$, $F(y, ., .)$ is diagonally upper $\mathcal{C}(y, .)$-convex (or, diagonally upper $\mathcal{C}(y, .)$-quasiconvex-like) in the second variable.

Then for any fixed $t \in D, y \in K$, the following are equivalent:

1) $F(y, t, x) \cap -(\mathcal{C}(y, t)\backslash\{0\}) = \emptyset$ for all $x \in D$;

2) $F(y, x, t) \cap -\mathcal{C}(y, x) \neq \emptyset$ for all $x \in D$.

Lemmas 2.4.2, 2.4.3 and 2.4.4 are similarly stated.

2.4.5.1. Pareto and weakly quasi-equilibrium problems type 1

Let $S : D \times K \to 2^D, T : D \times K \to 2^K$ and $G : K \times D \times D \to 2^Y$ be multivalued mapping with nonempty values, C be closed convex cones in Y. The upper (lower) Pareto quasi-equilibrium problems and upper (lower) weak quasi-equilibrium problems of type 1,respectively, are formed:

1. Find $\bar{x}, \bar{y} \in D \times K$ such that

$$\bar{x} \in S(\bar{x}, \bar{y}), \bar{y} \in T(\bar{x}, \bar{y}),$$
$$G(\bar{y}, \bar{x}, z) \not\subseteq -(\mathcal{C}(\bar{y}, \bar{x}) \setminus \{0\}) \text{ for all } z \in S(\bar{x}, \bar{y});$$

2. Find $\bar{x}, \bar{y} \in D \times K$ such that

$$\bar{x} \in S(\bar{x}, \bar{y}), \bar{y} \in T(\bar{x}, \bar{y}),$$
$$G(\bar{y}, \bar{x}, z) \cap -(\mathcal{C}(\bar{y}, \bar{x}) \setminus \{0\}) = \emptyset \text{ for all } z \in S(\bar{x}, \bar{y});$$

3. Find $\bar{x}, \bar{y} \in D \times K$ such that

$$\bar{x} \in S(\bar{x}, \bar{y}), \bar{y} \in T(\bar{x}, \bar{y}),$$
$$G(\bar{y}, \bar{x}, z) \nsubseteq -\text{int}\mathcal{C}(\bar{y}, \bar{x}) \text{ for all } z \in S(\bar{x}, \bar{y});$$

4. Find $\bar{x}, \bar{y} \in D \times K$ such that

$$\bar{x} \in S(\bar{x}, \bar{y}), \bar{y} \in T(\bar{x}, \bar{y}),$$
$$G(\bar{y}, \bar{x}, z) \cap -\text{int}\mathcal{C}(\bar{y}, \bar{x}) = \emptyset \text{ for all } z \in S(\bar{x}, \bar{y}).$$

Next, we present sufficient conditions for the existence of solutions to Pareto and weak quasi-equilibrium problems of type 1.

Theorem 2.4.2.(Lower Pareto quasi-equilibrium problems of type 1). *Let D, K are nonempty compact convex subsets, $G : K \times D \times D \to 2^Y$ be a multivalued mappings with nonempty valued and $G(y, x, x) \subseteq C$ for any $x \in D, y \in K$ satisfying the following conditions:*

i) S is continuous mappings with nonempty convex closed valued, T is lower semicontinuous mappings with nonempty convex closed valued;

ii) For any fixed $(x, y) \in D \times K$, $G(y, ., x) : D \to 2^Y$ is upper ideal $C-$ hemicontinuous;

iii) For any fixed $y \in K$, $G(y, ., .)$ is lower C-strong pseudomonotone;

iv) For any fixed $(x, y) \in K$, $G(y, x, .)$ is upper $C-$convex (or, upper $C-$ quasiconvex-like);

v) G is upper $C-$ continuous.

Then there exists $\bar{x} \in D, \bar{y} \in K$ such that

$$\bar{x} \in S(\bar{x}, \bar{y}), \bar{y} \in T(\bar{x}, \bar{y}),$$
$$G(\bar{y}, \bar{x}, z) \cap (-C \setminus \{0\}) = \emptyset \textit{ for all } z \in S(\bar{x}, \bar{y}).$$

Similarly, we have the results on the existence of the solutions to the rest problems (see Theorem 2.4.3, 2.4.4, 2.4.5).

2.4.5.2. Pareto and weakly quasi-equilibrium problems of type 2

In this section, we consider the mapping $G : D \times D \to 2^Y$ and cone mapping $\mathcal{C} : D \to 2^Y$ with nonempty values.

The upper (lower) Pareto quasi-equilibrium problems and upper (lower) weak quasi-equilibrium problems of type 2, respectively, are formed:

1. Find $\bar{x} \in D$ such that
   $\bar{x} \in P(\bar{x})$ and $G(\bar{x}, x) \nsubseteq -(\mathcal{C}(\bar{x}) \setminus \{0\}),$ for all $x \in P(\bar{x})$.

2. Find $\bar{x} \in D$ such that
   $\bar{x} \in P(\bar{x})$ and $G(\bar{x}, x) \cap -(\mathcal{C}(\bar{x}) \setminus \{0\}) = \emptyset,$ for all $x \in P(\bar{x})$.

3. Find $\bar{x} \in D$ such that
   $\bar{x} \in P(\bar{x})$ and $G(\bar{x}, x) \nsubseteq -\text{int}\mathcal{C}(\bar{x}),$ for all $x \in P(\bar{x})$.

4. Find $\bar{x} \in D$ such that
   $\bar{x} \in P(\bar{x})$ and $G(\bar{x}, x) \cap -\text{int}\mathcal{C}(\bar{x}) = \emptyset,$ for all $x \in P(\bar{x})$.

Theorem 2.4.9. (Lower Pareto quasi-equilibrium problem type 2.) *Let $D \subset X$ be a nonempty convex compact, $G : D \times D \to 2^Y$ be a multivalued mapping with nonempty values and $C \subseteq Y$ be a cone with $G(x, x) \subseteq C$ với mọi $x \in D$. Assume that:*

i) For any fixed $t \in D$, $G(., t) : D \to 2^Y$ is lower C-strong hemicontinuous;

ii) For any fixed $x \in D, y \in K$,

$$A = \{t \in D |\, G(x, t) \cap (-C) \neq \emptyset\} \text{ is closed in } D;$$

iii) G is lower $\mathcal{C}$-strong pseudomonotone;

iv) G is diagonally upper C–convex (or, diagonally upper C–quasiconvex-like) in the second variable.

Then there exists $\bar{x} \in D$ such that $\bar{x} \in P(\bar{x})$ and

$$G(\bar{x}, t) \cap (-C\backslash\{0\}) = \emptyset \text{ for all } t \in P(\bar{x}).$$

The other results are shown in Theorem 2.4.7, 2.4.8, 2.4.9. In Corollaries 2.4.9, 2.4.10, 2.4.11, 2.4.12, we will apply the results of Section 3 on the generalized vector variational inequality problems by replacing G with $F : D \times D \to 2^Y$, $F(x,t) = \langle G(x), \theta(x,t)\rangle$, $(x,t) \in D \times D$, where $G : D \to 2^{L(X,Y)}$.

Remark. If $Y = \mathbb{R}, \mathcal{C}(\bar{x}) \equiv \mathbb{R}^+$ and $G : D \to X^*$ is hemicontinuous and monotone mapping, $P(x) \equiv D, \theta(x,t) = t-x$, for all $x, t \in D$, then Corollary 2.4.9 becomes: There exists $\bar{x} \in D$ such that

$$\begin{aligned} &\langle G(\bar{x}), t - \bar{x}\rangle \geq 0, \\ &(\text{it is equivalent to } \ \langle G(t), \bar{x} - t\rangle \geq 0), \ \text{for all} \ \ t \in D. \end{aligned} \tag{2.9}$$

This is classic Stampacchia (Minty) variational inequality which we study in Chapter 4.

2.5. The stability of the solutions to generalized quasi-equilibrium problems

Let $X, Z, D, K, Y, \mathcal{C}$ be the same as in above sections. Let Λ, Γ, Σ be real topological Hausdorff spaces, the multivalued mappings $P_i : D \times \Lambda \to 2^D, i = 1, 2, Q : D \times D \times \Gamma \to 2^K$ and $F : K \times D \times D \times \Sigma \to 2^Y$. We get a generalized quasi-equilibrium problems dependent on parameters: Find $\bar{x} \in P_1(\bar{x}, \lambda)$ such that $0 \in F(y, \bar{x}, t, \mu)$ for all $t \in P_2(\bar{x}, \lambda), y \in Q(\bar{x}, t, \gamma)$.

For any $\lambda \in \Lambda, \mu \in \Gamma, \gamma \in \Sigma$, we set $E(\lambda) = \{x \in P_1(x, \lambda)\}$; $M(\lambda, \gamma, \mu) = \{x \in D \mid x \in E(\lambda) \text{ and } 0 \in F(y, x, t, \mu) \ \text{ for all } t \in P_1(x, \lambda), y \in Q(x, t, \gamma)\}$. Section 2.3 obtains the sufficient conditions for $M(\lambda, \gamma, \mu) \neq \emptyset$. Next, we show the sufficient conditions for the solution mappings characterized by stability: upper semicontinuity, lower semicontinuity to (λ, γ, μ).

Theorem 2.5.1. *Let $(\lambda_0, \gamma_0, \mu_0) \in \Lambda \times \Gamma \times \Sigma$. Suppose that:*

i) P_1 is an upper semicontinuous with compact valued mapping; P_2 is an lower semicontinuous mapping;

ii) Q is a lower semicontinuous with compact valued mapping;

iii) The set $A = \{(y, x, \lambda, \gamma, \mu) \mid x \in E(\lambda), 0 \in F(y, x, t, \gamma)$ với mọi $t \in P_2(x, \lambda), y \in Q(x, t, \mu)\}$ is closed.

Then M is upper semicontinuous and closed at $(\lambda_0, \gamma_0, \mu_0)$.

Theorem 2.5.2. *The mapping M be lower semicontinuous at $(\lambda_0, \gamma_0, \mu_0)$ if we have:*

i) E is lower semicontinuous at λ_0;

ii) Q is upper semicontinuous with compact values;

iii) P_2 *is a closed mapping;*

iv) The set $A = \{(y, x, t, \lambda, \gamma, \mu) \in D \times D \times D \times \Lambda \times \Gamma \times \Sigma \mid x \in P_1(x, \lambda), 0 \notin F(y, x, t, \lambda, \gamma, \mu), t \in P_2(x, \lambda), y \in Q(x, t, \mu)\}$ *is closed.*

SUMMARY OF CHAPTER 2

In this chapter, we prove the existence of the solutions to quasi-equilibrium generalized problems of type 2 and related problems, such as: undirected quasi-equilibrium problems, quasi-variational inclusions, quasi-variational related problems, Pareto and weak quasi-equilibrium problems (Sections 2.3, 2.4). Section 2.5 obtains the stability of the solutions to generalized quasi-equilibrium problems of type 2. These results were published in [3].

Chapter 3. MIXED PARETO QUASI-VARIATIONAL INCLUSION PROBLEMS

3.1. Introduction to problems

Throughout this chapter, except for special cases, we denote by X, Y, Y_1, Y_2, Z real locally convex Hausdorff topological vector spaces. Assume that $D \subset X, K \subset Z$ are nonempty subsets. and $C_i \subseteq Y_i, i = 1, 2$, are convex closed cones. 2^A denotes the collection of all subsets in the set A. Given multivalued mappings $S : D \times K \to 2^D, T : D \times K \to 2^K; P : D \to 2^D, Q : K \times D \to 2^K$ and $F_1 : K \times K \times D \to 2^{Y_1} F_2 : K \times D \times D \to 2^{Y_2}$, we consider the following problems:

1. Mixed upper-upper Pareto quasi-variational inclusion problem:

Find $(\bar{x}, \bar{y}) \in D \times K$ such that

$$\begin{aligned}
&\bar{x} \in S(\bar{x}, \bar{y}); \bar{y} \in T(\bar{x}, \bar{y});\\
&F_1(\bar{y}, v, \bar{x}) \not\subseteq F_1(\bar{y}, \bar{y}, \bar{x}) - (C_1 \setminus \{0\}), \quad \text{for all } v \in T(\bar{x}, \bar{y});\\
&F_2(y, \bar{x}, t) \not\subseteq F_2(y, \bar{x}, \bar{x}) - (C_2 \setminus \{0\}), \text{ for all } t \in P(\bar{x}), y \in Q(\bar{x}, t).
\end{aligned}$$

2. Mixed upper-lower Pareto quasi-variational inclusion problem:

Find $(\bar{x}, \bar{y}) \in D \times K$ such that

$$\begin{aligned}
&\bar{x} \in S(\bar{x}, \bar{y}); \bar{y} \in T(\bar{x}, \bar{y});\\
&F_1(\bar{y}, v, \bar{x}) \not\subseteq (F_1(\bar{y}, \bar{y}, \bar{x}) - (C_1 \setminus \{0\})), \text{ for all } v \in T(\bar{x}, \bar{y});\\
&F_2(y, \bar{x}, \bar{x}) \not\subseteq F_2(y, \bar{x}, t) + (C_2 \setminus \{0\})), \text{ for all}\, t \in P(\bar{x}), y \in Q(\bar{x}, t).
\end{aligned}$$

3. Mixed lower - upper Pareto quasi-variational inclusion problem:

Find $(\bar{x}, \bar{y}) \in D \times K$ such that

$$\begin{aligned}
&\bar{x} \in S(\bar{x}, \bar{y}); \bar{y} \in T(\bar{x}, \bar{y});\\
&F_1(\bar{y}, \bar{y}, \bar{x}) \not\subseteq F_1(\bar{y}, v, \bar{x}) + (C_1 \setminus \{0\})), \text{ for all } v \in T(\bar{x}, \bar{y});\\
&F_2(y, \bar{x}, t) \not\subseteq (F_2(y, \bar{x}, \bar{x}) - (C_2 \setminus \{0\})), \text{ for all}\, t \in P(\bar{x}), y \in Q(\bar{x}, t).
\end{aligned}$$

4. Mixed lower-lower Pareto quasi-variational inclusion problem:

Find $(\bar{x}, \bar{y}) \in D \times K$ such that

$$\bar{x} \in S(\bar{x}, \bar{y}); \bar{y} \in T(\bar{x}, \bar{y});$$
$$F_1(\bar{y}, \bar{y}, \bar{x}) \not\subseteq F_1(\bar{y}, v, \bar{x}) + (C_1 \setminus \{0\})), \text{ for all } v \in T(\bar{x}, \bar{y});$$
$$F_2(y, \bar{x}, \bar{x}) \not\subseteq F_2(y, \bar{x}, t) + (C_2 \setminus \{0\})), \text{ for all } t \in P(\bar{x}), y \in Q(\bar{x}, t).$$

These problems have emerged as a powerful tool for wide class of quasi-equilibrium, quasi- variational, quasi-optimization problems. There are few papers considering the mixed problems as above. But mostly these papers pay attention to one of type 1 and type 2 only. The purpose of this chapter is to study the existence of solutions to the mixed Pareto quasi-variational inclusion problems. Many problems in the vector optimization theory concerning multivalued mappings like quasi-equilibrium, quasi-variational inclusion, quasi-variational relation problems can be reduced to the form of these problems. Balaj and Luc also considered the mixed variational relations problems. But, their problem has no constraint multivalued mapping S. The solution set of this problem is found on whole set D. Our approach to prove the existence of solutions to these problems is unlike their methods. They used the finite intersection property of the mappings family which have KKM property with respect to a set-valued mapping, we use a lemma on empty intersection of two multivalued mappings to prove the existence of solutions to above mentioned problems.

3.2. Existence of solutions

Given multivalued mappings S, T, P, Q and $F_i, i = 1, 2$ with nonempty values as in Introduction section, we first prove the following theorem for the existence of solutions of the mixed upper-upper Pareto quasi-variational inclusion problem.

3.2.1. Upper-upper mixed Pareto quasi-variational inclusion problems

Theorem 3.2.1. *We assume that the following conditions hold:*

(i) D, K are nonempty convex compact subsets;

(ii) S is a multivalued with nonempty convex values and has open lower sections and T is a continuous multivalued mapping with nonempty

closed convex values and the subset $A = \{(x,y) \in D \times K | (x,y) \in S(x,y) \times T(x,y)\}$ *is closed;*

(iii) P *has open lower sections and* $P(x) \subseteq S(x,y)$ *for* $(x,y) \in A$. *For any fixed* $t \in D$, *the multivalued mapping* $Q(.,t) : D \to 2^K$ *is lower semi-continuous with compact values;*

(iv) *The multivalued mapping* F_1 *is a upper* $(-C_1)-$ *continuous and lower* C_1- *continuous mapping with nonempty weak compact values. For any fixed* $t \in D$ *the multivalued mapping* $F_2(.,.,t) : K \times D \to 2^{Y_2}$ *is a upper* $(-C_2)$*- continuous multivalued mapping with nonempty weak compact values and for any fixed* $y \in K$, *the multivalued mapping* $N_2 : K \times D \to 2^{Y_2}$ *defined by* $N_2(y,x) = F_2(y,x,x)$ *is lower* C_2-*continuous ;*

(v) *For any fixed* $(x,y) \in D \times K$, *the multivalued mapping* $F_1(y,.,x) : K \to 2^{Y_1}$ *is lower* C_1- *convex (or, lower* C_1-*quasi-convex-like) and any* $y \in K$ *the multivalued mapping* $F_2(y,.,.) : D \times D \to 2^{Y_2}$ *is diagonally lower* C_2*-convex in the second variable (or, diagonally lower* C_2*-quasi-convex-like in the second variable).*

Then there exists $(\bar{x},\bar{y}) \in D \times K$ *such that*

$$\bar{x} \in S(\bar{x},\bar{y}); \bar{y} \in T(\bar{x},\bar{y});$$
$$F_1(\bar{y},v,\bar{x}) \not\subseteq (F_1(\bar{y},\bar{y},\bar{x}) - (C_1 \setminus \{0\})), \text{ for all } v \in T(\bar{x},\bar{y});$$
$$F_2(y,\bar{x},t) \not\subseteq (F_2(y,\bar{x},\bar{x}) - (C_2 \setminus \{0\})), \text{ for all } t \in P(\bar{x}), y \in Q(\bar{x},t).$$

3.2.2. Upper-lower mixed Pareto quasi-variational inclusion problems

Theorem 3.2.1. *We assume that the following conditions hold:*

(i) D, K *are nonempty convex compact subsets;*

(ii) S *is a multivalued mapping with nonempty convex values and has open lower sections and* T *is a continuous multivalued mapping with nonempty closed convex values and the subset* $A = \{(x,y) \in D \times K | (x,y) \in S(x,y) \times T(x,y)\}$ *is closed;*

(iii) *P has open lower sections and* $P(x) \subseteq S(x,y)$ *for* $(x,y) \in A$. *For any fixed* $t \in D$, *the multivalued mapping* $Q(.,t) : D \to 2^K$ *is lower semi-continuous with compact values;*

(iv) *The multivalued mapping* F_1 *is a upper* $(-C_1)-$ *continuous and lower* C_1- *continuous mapping with nonempty weak compact values. For any fixed* $t \in D$ *the multivalued mapping* $F_2(.,.,t) : K \times D \to 2^{Y_2}$ *is a lower* $(-C_2)$*- continuous mapping with nonempty weak compact values and for any fixed* $y \in Y$, *the multivalued mapping* $N_2 : K \times D \to 2^{Y_2}$ *defined by* $N_2(y,x) = F_2(y,x,x)$ *is upper* C_2-*continuous;*

(v) *For any fixed* $(x,y) \in D \times K$, *the multivalued mapping* $F_1(y,.,x) : K \to 2^{Y_1}$ *is lower* C_1- *convex (or, lower* C_1-*quasi-convex-like) and any* $y \in K$ *the multivalued mapping* $F_2(y,.,.) : D \times D \to 2^{Y_2}$ *is diagonally upper* C_2*-convex in the second variable (or, diagonally upper* C_2*-quasi-convex-like in the second variable).*

Then there exists $(\bar{x},\bar{y}) \in D \times K$ *such that*

$$\bar{x} \in S(\bar{x},\bar{y}); \bar{y} \in T(\bar{x},\bar{y});$$
$$F_1(\bar{y},v,\bar{x}) \not\subseteq (F_1(\bar{y},\bar{y},\bar{x}) - (C_1 \setminus \{0\})), \text{ for all } v \in T(\bar{x},\bar{y});$$
$$F_2(y,\bar{x},\bar{x}) \not\subseteq F_2(y,\bar{x},t) + (C_2 \setminus \{0\})), \text{ for all} t \in P(\bar{x}), y \in Q(\bar{x},t).$$

3.2.3. Lower-upper mixed Pareto quasi-variational inclusion problems

Theorem 3.2.3. *We assume that the following conditions hold:*

(i) D, K *are nonempty convex compact subsets;*

(ii) *S is a multivalued with nonempty convex values and has open lower sections and T is a continuous multivalued mapping with nonempty closed convex values and the subset* $A = \{(x,y) \in D \times K | (x,y) \in S(x,y) \times T(x,y)\}$ *is closed;*

(iii) *P has open lower sections and* $P(x) \subseteq S(x,y)$ *for* $(x,y) \in A$. *For any fixed* $t \in D$, *the multivalued mapping* $Q(.,t) : D \to 2^K$ *is lower semi-continuous with compact values;*

(iv) *The multivalued mapping F_1 is a upper C_1- continuous and lower $(-C_1)-$ continuous mapping with nonempty weak compact values. For any fixed $t \in D$ the multivalued mapping $F_2(.,.,t) : K \times D \to 2^{Y_2}$ is a upper $(-C_2)$- continuous mapping with nonempty weak compact values and for any fixed $y \in Y$, the multivalued mapping $N_2 : K \times D \to 2^{Y_2}$ defined by $N_2(y,x) = F_2(y,x,x)$ is lower C_2-continuous;*

(v) *For any fixed $(x,y) \in D \times K$, the multivalued mapping $F_1(y,.,x) : K \to 2^{Y_1}$ is upper C_1- convex (or, upper C_1-quasi-convex-like) and any $y \in K$ the multivalued mapping $F_2(y,.,.) : D \times D \to 2^{Y_2}$ is diagonally lower C_2-convex in the second variable (or, diagonally lower C-quasi-convex-like in the second variable).*

Then there exists $(\bar{x},\bar{y}) \in D \times K$ such that

$$\bar{x} \in S(\bar{x},\bar{y}); \bar{y} \in T(\bar{x},\bar{y});$$
$$F_1(\bar{y},\bar{y},\bar{x}) \not\subseteq F_1(\bar{y},v,\bar{x}) + (C_1 \setminus \{0\})), \text{ for all } v \in T(\bar{x},\bar{y});$$
$$F_2(y,\bar{x},t) \not\subseteq (F_2(y,\bar{x},\bar{x}) - (C_2 \setminus \{0\})), \text{ for all} t \in P(\bar{x}), y \in Q(\bar{x},t).$$

3.2.4. Lower-lower mixed Pareto quasi-variational inclusion problems

Theorem 3.2.3. *We assume that the following conditions hold:*

(i) *D, K are nonempty convex compact subsets;*

(ii) *S is a multivalued with nonempty convex values and has open lower sections and T is a continuous multivalued mapping with nonempty closed convex values and the subset $A = \{(x,y) \in D \times K | (x,y) \in S(x,y) \times T(x,y)\}$ is closed;*

(iii) *P has open lower sections and $P(x) \subseteq S(x,y)$ for $(x,y) \in A$. For any fixed $t \in D$, the multivalued mapping $Q(.,t) : D \to 2^K$ is lower semi-continuous with compact values;*

(iv) *The multivalued mapping F_1 is a upper C_1- continuous and lower $(-C_1)-$ continuousmapping with nonempty weak compact values. For any fixed $t \in D$ the multivalued mapping $F_2(.,.,t) : K \times D \to 2^{Y_2}$ is a lower $(-C_2)$- continuous mapping with nonempty weak compact values* *and for any fixed* $y \in Y$*, the multivalued mapping* $N_2 : K \times D \to 2^{Y_2}$ *defined by* $N_2(y,x) = F_2(y,x,x)$ *is upper* C_2-*continuous ;*

(v) *For any fixed* $(x,y) \in D \times K$*, the multivalued mapping* $F_1(y,.,x) : K \to 2^{Y_1}$ *is upper* C_1- *convex (or, upper* C_1-*quasi-convex-like) and any* $y \in K$ *the multivalued mapping* $F_2(y,.,.) : D \times D \to 2^{Y_2}$ *is diagonally upper* C_2*-convex in the second variable (or, diagonally upper* C*-quasi-convex-like in the second variable).*

Then there exists $(\bar{x}, \bar{y}) \in D \times K$ *such that*

$$
\begin{aligned}
&\bar{x} \in S(\bar{x},\bar{y}); \bar{y} \in T(\bar{x},\bar{y});\\
&F_1(\bar{y},\bar{y},\bar{x}) \not\subseteq F_1(\bar{y},v,\bar{x}) + (C_1 \setminus \{0\})), \textit{ for all } v \in T(\bar{x},\bar{y});\\
&F_2(y,\bar{x},\bar{x}) \not\subseteq F_2(y,\bar{x},t) + (C_2 \setminus \{0\})), \textit{ for all } t \in P(\bar{x}), y \in Q(\bar{x},t).
\end{aligned}
$$

We assume that all the hypotheses of Theorem 3.3.1–3.3.4 are satisfied except for (i) and (iii) (respectively) replaced by

(i') S is a lower semi-continuous multivalued mapping with nonempty convex values;

(iii') P is lower semi-continuous and $P(x) \subseteq S(x,y)$ for all $x \in S(x,y), y \in T(x,y)$ and the subset $A = \{(x,y) \in D \times K | (x,y) \in S(x,y) \times T(x,y)\}$ is closed.

Then the conclusions of these theorems are also true.

3.3. Some problems concern to mixed Pareto quasivariational inclusion problems

Given multivalued mappings S, T and $F_i, i = 1,2$ with nonempty values as in Introduction, Section 3.3, we are interested in the related problems: systems of quasi-variational inclusion problems of types 1 and 2.

1. System of two upper Pareto quasi-variational inclusion problems of type 1. Find $(\bar{x}, \bar{y}) \in D \times K$ such that

$$\bar{x} \in S(\bar{x}, \bar{y}); \bar{y} \in T(\bar{x}, \bar{y});$$
$$F_1(\bar{y}, v, \bar{x}) \not\subseteq F_1(\bar{y}, \bar{y}, \bar{x}) - (C_1 \setminus \{0\}), \quad \text{for all} \quad v \in T(\bar{x}, \bar{y});$$
$$F_2(\bar{y}, \bar{x}, t) \not\subseteq F_2(\bar{y}, \bar{x}, \bar{x}) - (C_2 \setminus \{0\}), \text{for all } t \in S(\bar{x}, \bar{y}).$$

2. System of upper and lower quasi-variational inclusion problems of type 1. Find $(\bar{x}, \bar{y}) \in D \times K$ such that

$$\bar{x} \in S(\bar{x}, \bar{y}); \bar{y} \in T(\bar{x}, \bar{y});$$
$$F_1(\bar{y}, v, \bar{x}) \not\subseteq F_1(\bar{y}, \bar{y}, \bar{x}) - (C_1 \setminus \{0\}), \quad \text{for all} \quad v \in T(\bar{x}, \bar{y});$$
$$(F_2(\bar{y}, \bar{x}, \bar{x}) \not\subseteq F_2(\bar{y}, \bar{x}, t) + (C_2 \setminus \{0\}), \text{for all } t \in S(\bar{x}, \bar{y}).$$

3. System of two lower quasi-variational inclusion problems of type 1. Find $(\bar{x}, \bar{y}) \in D \times K$ such that

$$\bar{x} \in S(\bar{x}, \bar{y}); \bar{y} \in T(\bar{x}, \bar{y});$$
$$F_1(\bar{y}, \bar{y}, \bar{x}) \not\subseteq F_1(\bar{y}, v, \bar{x}) + (C_1 \setminus \{0\}), \quad \text{for all} \quad v \in T(\bar{x}, \bar{y});$$
$$F_2(\bar{y}, \bar{x}, \bar{x}) \not\subseteq F_2(\bar{y}, \bar{x}, t) + (C_2 \setminus \{0\}), \text{for all } t \in S(\bar{x}, \bar{y}).$$

Theorems 3.3.4, 3.3.5 consider mixed Pareto quasi-equilibrium problems by adding $F_1(y, y, x) \subseteq C_1$ and $F_2(y, x, x) \subseteq C_2$ for all $(x, y) \in D \times K$.

SUMMARY OF CHAPTER 3

In Chapter 3, we introduce mixed Pareto quasi-variational inclusion problems and show some sufficient conditions on the existence of their solutions. As special cases, we obtain several results for different mixed Pareto quasi-equilibrium problems, mixed Pareto quasi- optimization problems and also mixed weak quasi-variational inclusion problems etc.

Chapter 4. IMPLICIT ITERATION METHODS FOR FINDING SOLUTIONS TO VARIATIONAL INEQUALITIES

4.1.Introduction to problems

Variational inequalities were initially studied by Stampacchia and from then on have been widely investigated, they have covered as diverse disciplines like partial differential equations, optimal control, optimization, mathematical programming, mechanics, and finance. In this chaper, we introduce a new implicit iteration method for finding a solution for a variational inequality involving Lipschitz continuous and strongly monotone mapping over the set of common fixed points of a finite family of nonexpansive mappings on Hilbert spaces. The problem is formulated as finding a point $\overline{x} \in D$ such that

$$\begin{aligned} &\langle G(\overline{x}), x - \overline{x}\rangle \geq 0, \forall x \in D, \\ &D = \bigcap_{i=1}^{n} \text{Fix}(T_i), \end{aligned} \tag{4.1}$$

where $N \in \mathbb{N}, T_i : X \to X, i = 1, 2, ..., N$, are nonexpansive mappings. It is the problem: finding solutions of variational inequalities on interrelationship of the fixed point sets of nonexpansive mappings $T_i, i = 1, 2, ..., N$. We define the mappings $P_1, P_2 : D \to D$,

$$P_1(x) = \{t \in D : \langle T_i(x) - t, x - y\rangle \geq 0, \text{ for all } i = 1, 2, ..., n, y \in D\},$$

$$P_2(x) = \{t \in D : \langle T_i(x) - t, x - y\rangle > 0, \text{ for all } i = 1, 2, ..., n, y \in D\}$$

and $F(y, x, t) = \langle G(x), y - t\rangle - \mathbb{R}_+, y, x, t \in D$. Set $K = D, Q(x, t) = D$, we consider generalized quasi-equilibrium problem of type 2: Tìm $\bar{x} \in D, \bar{x} \in P_1(\bar{x}),\ 0 \in F(y, \bar{x}, t),$ for all $t \in P_2(\bar{x}), y \in Q(\bar{x}, t)$. If $\bar{x}$ is a solution of generalized quasi-equilibrium type 2 problem, then we get $\bar{x} \in P_1(\bar{x})$. That means

$$\langle T_i(\bar{x}) - \bar{x}, \bar{x} - y\rangle \geq 0, \text{ for all } y \in D, i = 1, 2, ..., n.$$

Let $y = T_i(\bar{x}), i = 1, 2, ..., n$, it follows that $\langle T_i(\bar{x}) - \bar{x}, \bar{x} - T_i(\bar{x})\rangle \geq 0$, or $\| T_i(\bar{x}) - \bar{x} \| \leq 0$. We have $T_i(\bar{x}) = \bar{x}, i = 1, 2, ..., n$. With $0 \in F(y, \bar{x}, t),$ for all $t \in P_2(\bar{x}), y \in Q(\bar{x}, t)$, we have $\langle G(\bar{x}), y - \bar{x}\rangle \geq 0,$ for all $y \in D$. So, $\bar{x}$ is a solution of variational inequality problem (4.1).

In contrast, if $\bar{x}$ is a solution of variational inequality on interrelationship of the fixed point sets of nonexpansive mappings $T_i, i = 1, 2, ..., N$, then $\bar{x}$ is a solution of generalized quasi-equilibrium of type 2.

Theorems 4.1.1, 4.1.2 introduce the implicit iteration process for finding an element $p \in \cap_{i=1}^{N} Fix(T_i)$ (Xu H.K. -Ori R.G., 2001 and Zheng L.C. - Yao J.C., 2006). These results are weak convergence. Clearly, from $\sum_{k=1}^{\infty} \lambda_k < \infty$ we have that $\lambda_k \to 0$ as $k \to \infty$. In Section 4.2, we propose the implicit iteration algorithms which converges strongly to the solution of (4.1) without the condition $\sum_{k=1}^{\infty} \lambda_k < \infty$.

4.2. An implicit iteration methods on the set of common fixed points for a finite family of nonexpansive mappings in Hilbert spaces.

Let X be a Hilbert space, the mapping $G : X \to X$, the parameters $\mu \in (0, 2\eta/L^2)$ and $t \in (0,1), \{\lambda_t\}, \{\beta_t^i\} \subset (0,1)$, such that

$$\begin{aligned} &\lambda_t \to 0, \text{if} \quad t \to 0 \quad \text{and} \\ &0 < \liminf_{t\to 0} \beta_t^i \leq \limsup_{t\to 0} \beta_t^i < 1, 2, \quad i = 1, 2, \cdots, N. \end{aligned} \tag{4.5}$$

The net $\{x_t\}$ defined by

$$x_t = T^t x_t, \quad T^t := T_0^t T_N^t ... T_1^t, \quad t \in (0,1), \tag{4.6}$$

where $T_i^t : X \to X$,

$$\begin{aligned} &T_i^t x = (1 - \beta_t^i)x + \beta_t^i T_i x, \quad i = 1, 2, \cdots, N, \\ &T_0^t y = (I - \lambda_t \mu G)y, \ x, y \in X. \end{aligned} \tag{4.7}$$

Theorem 4.2.1. *Let X be a real Hilbert space and $G : X \to X$ be a mapping such that for some constants $L, \eta > 0$, F is L-Lipschitz continuous and η-strongly monotone. Let $\{T_i\}_{i=1}^N$ be N nonexpansive self-maps of X such that $\mathcal{D} = \cap_{i=1}^N Fix(T_i) \neq \emptyset$. Let $\mu \in (0, 2\eta/L^2)$ and let $t \in (0,1), \{\lambda_t\}, \{\beta_t^i\} \subset (0,1)$, such that*

$$\lambda_t \to 0, \textit{as} \quad t \to 0 \quad \textit{and} \quad 0 < \liminf_{t\to 0} \beta_t^i \leq \limsup_{t\to 0} \beta_t^i < 1, \quad i = 1, 2, \cdots, N.$$

Then, the net $\{x_t\}$ defined by (4.5)-(4.7) converges strongly to the unique element $\bar{x}$ in (4.2) (with $D = X$).

Next, let $\alpha_i \in [\gamma_i, 1)$ be fixed real, $\{S_i\}_{i=1}^N$ be N mappings γ_i-strictly pseudocontractive in X. Theorem 4.2.2 extend our result to the case $\mathcal{D} = \cap_{i=1}^N Fix(S_i)$.

Theorem 4.2.2. *Let X be a real Hilbert space and $G : X \to X$ be a mapping such that for some constants $L, \eta > 0$, G is L-Lipschitz continuous and η-strongly monotone. Let $\{S_i\}_{i=1}^N$ be N γ_i-strictly pseudocontractive self-maps of X such that $\mathcal{D} = \cap_{i=1}^N Fix(S_i) \neq \emptyset$. Let $\alpha_i \in [\gamma_i, 1), \mu \in (0, 2\eta/L^2)$ and let $t \in (0,1), \{\lambda_t\}, \{\beta_t^i\} \subset (0,1)$, such that*

$$\lambda_t \to 0, \textit{as} \quad t \to 0 \quad \textit{and} \quad 0 < \liminf_{t\to 0} \beta_t^i \leq \limsup_{t\to 0} \beta_t^i < 1, \quad i = 1, \cdots, N.$$

Cho $\alpha_i \in [\gamma_i, 1), \mu \in (0, 2\eta/L^2)$ *và cho* $t \in (0,1), \{\lambda_t\}, \{\beta_t^i\} \subset (0,1)$, *như trong Định lý 4.2.1. Then, the net* $\{x_t\}$ *defined by*

$$x_t = \tilde{T}^t x_t, \quad \tilde{T}^t := T_0^t \tilde{T}_N^t ... \tilde{T}_1^t, \quad t \in (0,1),$$

where $\tilde{T}_i^t$, *for* $i = 1, 2, \cdots, N$, *are defined by*

$$x_t = \tilde{T}^t x_t, \quad \tilde{T}^t := T_0^t \tilde{T}_N^t ... \tilde{T}_1^t, \quad t \in (0,1),$$

and $T_0^t x = (I - \lambda_t \mu G)x$, *converges strongly to the unique element* $\bar{x}$ *in (4.2).*

Furthermore, we are interested in variational inequality problems on the common fixed set of infinite family of nonexpansive mappings in Banach spaces. That result are published in [5].

SUMMARY OF CHAPTER 4

In this chapter, Section 4.2, we introduce a new implicit iteration method for finding a solution for a variational inequality involving Lipschitz continuous and strongly monotone mapping over the set of common fixed points for a finite family of nonexpansive mappings on Hilbert spaces. Beside, for finding solutions of variational inequality problems in Banach spaces, we show modified viscosity approximation methods with weak contraction mapping for an infinite family of nonexpansive mappings. The results are published in [2] and [5].

Summary of dissertation and open issues

Summary of dissertation

1) The generalized quasi-equilibrium problem of type 2 is formulated.

2) Some sufficient conditions on the existence solutions of quasi-equilibrium problem of type 2 are shown.

3) In special cases, the dissertation shows several results on the existence of solutions to another problems in the vector optimization theory concerning multivalued mappings.

4) The dissertation introduces mixed Pareto quasi-variational inclusion problems and show some sufficient conditions on the existence of their solutions.

5) The dissertation introduces a new implicit iteration method for finding a solution for a variational inequality.

The open issues

1) Study about the application of the results in economic problems.

2) Continue to study the upper (lower) semicontinuity and Holder of the solutions of the general quasi-equilibrium problems.

3) Search for algorithms solving general quasi-equilibrium problems in a few special cases.

4) Study the problems in the case that collectives D, K are not compact, only convex and closed.