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APPROXIMATIVE METHODS FOR FIXED POINTS OF NONEXPANSIVE MAPPINGS AND SEMIGROUPS

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Introduction

Fixed point theory has many applications in variety of mathematical branches. Many problems arising in different areas of mathematics reduce to the problem of finding fixed points of a certain mapping such as integral equations, differential equations, or the problem of existence of variational inequalities, equilibrium problems, optimization and approximation theory. These theory is the basic for the development of fixed points of contraction mapping in finite dimensional spaces to many other classes of mappings, for instance Lipschitzian mappings, pseudocontractive mappings in Hilbert spaces and Banach spaces.

Theory of fixed point problems, including existence and methods for approximation of fixed points, has been considered by many well-known mathematicians such as Brower E., Banach S., Bauschke H. H., Moudafi A., Xu H. K., Schauder J., Browder F. E., Ky Fan K., Kirk W. A., Nguyen Buong, Phm Ky Anh, Le Dung Muu, etc .... Recently, problem of finding common fixed points of nonexpansive mappings and nonexpansive semigroups hosts a lots of research works in the field of nonlinear analysis with many publications of Vietnamese authors. For instance, Pham Ky Anh, Cao Van Chung (2014) "Parallel Hybrid Methods for a Finite Family of Relatively Nonexpansive Mappings", *Numerical Functional Analysis and Optimization.*, 35, pp. 649-664; P. N. Anh (2012) "Strong convergence theorems for nonexpansive mappings and Ky Fan inequalities", *J. Optim. Theory Appl.*, 154, pp. 303-320; P. N. Anh, L. D. Muu (2014) "A hybrid subgradient algorithm for nonexpansive mappings and equilibrium problems", *Optim. Lett.*, 8, pp. 727-738; Nguyen Thi Thu Thuy: (2013) "A new hybrid method for variational inequality and fixed point problems", *Vietnam. J. Math.*, 41, pp. 353-366, (2014) "Hybrid Mann-Halpern iteration methods for finding fixed points involving asymptotically nonexpansive mappings and semigroups", *Vietnam. J. Math.*, Volume 42, Issue 2, pp. 219-232, "An iterative method for equilibrium, variational inequality, and fixed point problems for a nonexpansive semigroup in Hilbert spaces", *Bull. Malays. Math. Sci. Soc.*,Volume 38, Issue 1, pp. 113-130, (2015) "A strongly strongly convergent shrinking descent-like Halpern's method for monotone variational inequaliy and fixed point problems", *Acta. Math. Vietnam.*, Volume 39, Issue 3, pp. 379-391; Nguyen Thi Thu Thuy, Pham Thanh Hieu

(2013) "Implicit Iteration Methods for Variational Inequalities in Banach Spaces", *Bull. Malays. Math. Sci. Soc.*, (2) 36(4), pp. 917-926; Duong Viet Thong: (2011), "An implicit iteration process for nonexpansive semi-groups", *Nonlinear Anal.*, 74, pp. 6116-6120, (2012) "The comparison of the convergence speed between picard, Mann, Ishikawa and two-step iterations in Banach spaces", *Acta. Math. Vietnam.*, Volume 37, Number 2, pp. 243-249, "Viscosity approximation method for Lipschitzian pseudo-contraction semigroups in Banach spaces", *Vietnam. J. Math.*, 40:4, pp. 515-525, etc....

It is worth mentioning some well-known types of iterative procedures, Krasnosel'skii iteration, Mann iteration, Halpern iteration, and Ishikawa one, etc.... These algorithms have been studied extensively and are still the focus of a host of research works.

Let C be a nonempty closed convex subset in a real Hilbert space H and let $T : C \to H$ be a nonexpansive mapping. Nakajo and Takahashi introduced the hybrid Mann's iteration method

$$\begin{cases} x_0 \in C \text{ any element}, \\ y_n = \alpha_n x_n + (1-\alpha_n)T(x_n), \\ C_n = \{z \in C : \ \|y_n - z\| \leq \|x_n - z\|\}, \\ Q_n = \{z \in C : \ \langle x_n - z, x_0 - x_n\rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n}(x_0), \quad n \geq 0, \end{cases} \tag{0.1}$$

where $\{\alpha_n\} \subset [0, a]$ for some $a \in [0, 1)$. They showed that $\{x_n\}$ defined by (0.1) converges strongly to $P_{F(T)}(x_0)$ as $n \to \infty$.

Moudafi A. proposed a viscosity approximation method

$$\begin{cases} x_0 \in C \text{ any element}, \\ x_n = \dfrac{1}{1+\lambda_n}T(x_n) + \dfrac{\lambda_n}{1+\lambda_n}f(x_n), \quad n \geq 0, \end{cases} \tag{0.2}$$

and

$$\begin{cases} x_0 \in C \text{ any element}, \\ x_{n+1} = \dfrac{1}{1+\lambda_n}T(x_n) + \dfrac{\lambda_n}{1+\lambda_n}f(x_n), \quad n \geq 0, \end{cases} \tag{0.3}$$

$f : C \to C$ be a contraction with a coefficient $\tilde{\alpha} \in [0, 1)$.

Alber Y. I. introduced a hybrid descent-like method

$$x_{n+1} = P_C(x_n - \mu_n[x_n - Tx_n]), \quad n \geq 0, \tag{0.5}$$

and proved that if $\{\mu_n\} : \mu_n > 0, \mu_n \to 0$, as $n \to \infty$ and $\{x_n\}$ is bounded.

Nakajo and Takahashi also introduced an iteration procedure as follows:

$$\begin{cases} x_0 \in C \text{ any element}, \\ y_n = \alpha_n x_n + (1-\alpha_n)\frac{1}{t_n}\int_0^{t_n} T(s)x_n ds, \\ C_n = \{z \in C : \|y_n - z\| \le \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_n - x_0, z - x_n\rangle \ge 0\}, \\ x_{n+1} = P_{C_n \cap Q_n}(x_0), \quad n \ge 0, \end{cases} \tag{0.6}$$

where $\{\alpha_n\} \in$ [0,a] for some $a \in$ [0,1) and $\{t_n\}$ is a positive real number divergent sequence. Further, in 2008, Takahashi, Takeuchi and Kubota proposed a simple variant of (0.6) that has the following form:

$$\begin{cases} x_0 \in H,\ C_1 = C,\ x_1 = P_{C_1}x_0, \\ y_n = \alpha_n x_n + (1-\alpha_n)T_n x_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \le \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}}x_0, \quad n \ge 1. \end{cases} \tag{0.7}$$

They showed that if $0 \le \alpha_n \le a < 1, 0 < \lambda_n < \infty$ for all $n \ge 1$ and $\lambda_n \to \infty$, then $\{x_n\}$ converges strongly to $u_0 = P_{\mathcal{F}}x_0$. At the time, Saejung considered the following analogue without Bochner integral:

$$\begin{cases} x_0 \in H,\ C_1 = C,\ x_1 = P_{C_1}x_0, \\ y_n = \alpha_n x_n + (1-\alpha_n)T(t_n)x_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \le \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}}x_0, \quad n \ge 0, \end{cases} \tag{0.8}$$

where $0 \le \alpha_n \le a < 1$, $\liminf_n t_n = 0, \limsup_n t_n > 0$, and $\lim_n(t_{n+1} - t_n) = 0$ and they proved that $\{x_n\}$ converges strongly to $u_0 = P_{\mathcal{F}}x_0$.
Recently, Nguyen Buong, introduced a new approach in order to replace closed and convex subsets C_n and Q_n by half spaces. Inspired by Nguyen Buong's idea, in this dissertation we propose some modification to approximate fixed points of nonexpansive mapppings and nonexpansive semigroups in Hilbert spaces.

Chapter 1

Preliminaries

1.1. Approximative Methods For Fixed Points of Nonexpansive Mappings

1.1.1. On Some Properties of Hilbert Spaces

Definition 1.1 Let H be a real Hilbert space. A sequence $\{x_n\}$ is called strong convergence to an element $x \in H$, denoted by $x_n \to x$, if $||x_n - x|| \to 0$ as $n \to \infty$.

Definition 1.2 A sequence $\{x_n\}$ is called weak convergence to an element $x \in H$, denoted by $x_n \rightharpoonup x$, if $\langle x_n, y\rangle \to \langle x, y\rangle$ as $n \to \infty$ vi mi $y \in H$.

1.1.2. Methods For Approximation of Fixed Points of Nonexpansive Mappings

Statement of problem: Let C be a nonempty, closed and convex subset in a Hilbert space H, $T : C \to C$ be a nonexpansive mapping. Find $x^* \in C :\ T(x^*) = x^*$.

Mann Iteration

In 1953, Mann W. R. introduced the following iteration

$$\begin{cases} x_0 \in C \text{ any element}, \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n, \quad n \geq 0, \end{cases} \tag{1.1}$$

and proved that, if $\{\alpha_n\}$ is chosen such that $\sum_{n=1}^{\infty} \alpha_n(1 - \alpha_n) = \infty$, then $\{x_n\}$ defined by (1.1) weakly convergent to a fixed point of mapping T.

Halpern Iteration

In 1967, Halpern B. considered the following method:

$$\begin{cases} x_0 \in C \text{ any element}, \\ x_{n+1} = \alpha_n u + (1-\alpha_n)Tx_n, \quad n \geq 0 \end{cases} \tag{1.2}$$

where $u \in C$ and $\{\alpha_n\} \subset (0,1)$ and proved that sequence (1.2) is strong convergent to a fixed point of nonexpansive mapping T with condition $\alpha_n = n^{-\alpha}, \alpha \in (0,1)$.

Ishikawa Iteration

In 1974, Ishikawa S. introduced a new iterative method as follows.

$$\begin{cases} x_1 \in C, \\ y_n = \beta_n x_n + (1-\beta_n)T(x_n), \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n)T(y_n), \quad n \geq 0, \end{cases} \tag{1.3}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences of real numbers belonging in interval $[0,1]$.

Vicosity Approximation

Moudafi A. (2000) "Viscosity approximation methods for fixed-point problems", *J. Math. Anal. Appl.*, 241, pp. 46-55., proposed a new method for finding common fixed points of nonexpansive mapppings in Hilbert spaces called viscosity approximation method and proved the following result.

Theorem 1.2 *Let C be a nonempty closed convex subset of a Hilbert space H and let T be a nonexpansive self-mapping of C such that $F(T) \neq \emptyset$. Let f be a contraction of C with a constant $\tilde{\alpha} \in [0,1)$ and let $\{x_n\}$ be a sequence generated by: $x_1 \in C$ and*

$$x_n = \frac{\lambda_n}{1+\lambda_n} f(x_n) + \frac{1}{1+\lambda_n} Tx_n, \quad n \geq 1, \tag{1.4}$$

$$x_{n+1} = \frac{\lambda_n}{1+\lambda_n} f(x_n) + \frac{1}{1+\lambda_n} Tx_n, \quad n \geq 1, \tag{1.5}$$

where $\{\lambda_n\} \subset (0,1)$ satisfies the following conditions:

(L1) $\lim\limits_{n\to\infty} \lambda_n = 0$;

(L2) $\sum\limits_{n=1}^{\infty} \lambda_n = \infty$;

(L3) $\lim\limits_{n\to\infty}\left|\frac{1}{\lambda_{n+1}}-\frac{1}{\lambda_n}\right|=0.$

Then, $\{x_n\}$ *defined by (1.5) converges strongly to* $p^* \in F(T)$*, where* $p^* = P_{F(T)}f(p^*)$ *and* $\{x_n\}$ *defined by (1.4) converges to* p^* *only under condition* (L1).

Hybrid Steepest Descent Method

Alber Ya. I. proposed the following descent-like method

$$x_{n+1} = P_C(x_n - \mu_n[x_n - Tx_n]), \quad n \geq 0, \tag{1.6}$$

and proved that: if $\{\mu_n\} : \mu_n \to 0$, as $n \to \infty$ and $\{x_n\}$ is bounded, then:

(a) there exists a weak accumulation point $\tilde{x} \in C$ of $\{x_n\}$;

(b) all weak accumulation points of $\{x_n\}$ belong to $F(T)$; and

(c) if $F(T)$ is a singleton, then $\{x_n\}$ converges weakly to $\tilde{x}$.

1.2. Nonexpansive Semigroups And Some Approximative Methods For Finding Fixed Points of Nonexpansive Semigroups

In 2010, Nguyen Buong (2010) "Strong convergence theorem for nonexpansive semigroups in Hilbert space", *Nonlinear Anal.*, 72(12), pp. 4534-4540, introduced a result as a improvement of some results of Nakajo K., Takahashi W. and Saejung S. stating in the following theorem.

Theorem 1.5 *Let* C *be a nonempty, closed and convex subset of a Hilbert space* H *and let* $\{T(t) : t \geq 0\}$ *be a nonexpansive semigroup on* C *with* $\mathcal{F} = \cap_{t\geq 0} F(T(t)) \neq \emptyset$*. Define a sequence* $\{x_n\}$ *by*

$$\begin{cases} x_0 \in H \text{ any element}, \\ y_n = \alpha_n x_n + (1-\alpha_n)T_n P_C(x_n), \\ \alpha_n \in (a,b],\ 0 < a < b < 1, \\ H_n = \{z \in H : \|z - y_n\| \leq \|z - x_n\|\}, \\ W_n = \{z \in H : \langle z - x_n, x_0 - x_n\rangle \leq 0\}, \\ x_{n+1} = P_{H_n\cap W_n}(x_0), \end{cases} \tag{1.9}$$

If $\liminf_{n\to\infty} t_n = 0$; $\limsup_{n\to\infty} t_n > 0$; $\lim_{n\to\infty}(t_{n+1} - t_n) = 0$*, then sequence* $\{x_n\}$ *defined (1.9) is strongly convergent to* $z_0 = P_{\mathcal{F}}(x_0)$.

Chapter 2

Approximative Methods For Fixed Points of Nonexpansive Mappings

2.1. Modified Viscosity Approximation

We propose some new modifications of (0.2) that are the implicit algorithm

$$x_n = T^n x_n,\ T^n := T_1^n T_0^n \quad \text{and} \quad T^n := T_0^n T_1^n,\ n \in (0,1), \tag{2.1}$$

where T_i^n are defined by

$$\begin{aligned} T_0^n &= (1-\lambda_n\mu)I + \lambda_n\mu f,\\ T_1^n &= (1-\beta_n)I + \beta_n T, \end{aligned} \tag{2.2}$$

where f is a contraction with a constant $\tilde{\alpha} \in [0,1)$, $\mu \in (0, 2(1-\tilde{\alpha})/(1+\tilde{\alpha})^2)$ and the parameters $\{\lambda_n\} \subset (0,1)$ and $\{\beta_n\} \subset (\alpha, \beta)$ for all $n \in (0,1)$ and some $\alpha, \beta \in (0,1)$ satisfying the following condition: $\lambda_n \to 0$ as $n \to 0$.

Theorem 2.1 *Let C be a nonempty closed convex subset of a real Hilbert space H and $f : C \to C$ be a contraction with a coefficient $\tilde{\alpha} \in [0,1)$. Let T be a nonexpansive self-mapping of C such that $F(T) \neq \emptyset$. Let $\mu \in (0, 2(1-\tilde{\alpha})/(1+\tilde{\alpha})^2)$. Then, the net $\{x_n\}$ defined by (2.1), (2.2) converges strongly to the unique element $p^* \in F(T)$ in $\langle (I-f)(p^*), p^* - p\rangle \leq 0,\ \forall p \in F(T)$.*

Next, we give two improvements of explicit method (0.3) in the form as follows

$$\begin{cases} x_1 \in C \text{ any element},\\ y_n = (1-\lambda_n\mu)x_n + \lambda_n\mu f(x_n),\\ x_{n+1} = (1-\gamma_n)x_n + \gamma_n T y_n,\ n \geq 1, \end{cases} \tag{2.8}$$

where $\{\lambda_n\} \subset (0,1)$, $\{\gamma_n\} \subset (\alpha,\beta)$, vi $\alpha,\beta \in (0,1)$ and

$$\begin{cases} x_1 \in C \text{ any element,} \\ y_n = (1-\beta_n)x_n + \beta_n T x_n, \\ x_{n+1} = (1-\gamma_n)x_n + \gamma_n[(1-\lambda_n\mu)y_n + \lambda_n\mu f(y_n)], \end{cases} \tag{2.9}$$

where $\{\beta_n\} \subset (\alpha,\beta)$.

Theorem 2.2 *Let C be a nonempty closed convex subset of a real Hilbert space H, $f : C \to C$ be a contraction with a coefficient $\tilde{\alpha} \in [0,1)$ and let T be a nonexpansive self-mapping of C such that $F(T) \neq \emptyset$. Assume that $\mu \in (0, 2(1-\tilde{\alpha})/(1+\tilde{\alpha})^2)$, $\{\lambda_k\} \subset (0,1)$ satisfying conditions (L1) $\lim_{n\to\infty} \lambda_n = 0$ and (L2) $\sum_{n=1}^{\infty} \lambda_n = \infty$ and $\{\gamma_n\} \subset (\alpha,\beta)$ for some $\alpha,\beta \in (0,1)$. Then, the sequence $\{x_k\}$ defined by (2.8) converges strongly to the unique element $p^* \in F(T)$ in $\langle (I-f)(p^*), p^* - p\rangle \leq 0$, $\forall p \in F(T)$. The same reult is guaranteed for $\{x_n\}$ defined by (2.9), if in addition, $\{\beta_n\} \subset (\alpha,\beta)$ satisfies the following condition: $|\beta_{n+1} - \beta_n| \to 0$ as $n \to \infty$.*

2.2. Modified Mann-Halpern Method

We proposed new methods in the following form:

$$\begin{cases} x_0 \in H \text{ any element,} \\ z_n = \alpha_n P_C(x_n) + (1-\alpha_n) P_C T P_C(x_n), \\ y_n = \beta_n x_0 + (1-\beta_n) P_C T z_n, \\ H_n = \{z \in H : \|y_n - z\|^2 \leq \|x_n - z\|^2 \\ \qquad + \beta_n(\|x_0\|^2 + 2\langle x_n - x_0, z\rangle)\}, \\ W_n = \{z \in H : \langle x_n - z, x_0 - x_n\rangle \geq 0\}, \\ x_{n+1} = P_{H_n \cap W_n}(x_0), \quad n \geq 0. \end{cases} \tag{2.13}$$

We have the following theorem:

Theorem 2.3 *Let C be a nonempty closed convex subset in a real Hilbert space H and let $T : C \to H$ be a nonexpansive mapping such that $F(T) \neq \emptyset$. Assume that $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in [0,1] such that $\alpha_n \to 1$ and $\beta_n \to 0$. Then, the sequences $\{x_n\}, \{y_n\}$ and $\{z_n\}$*

defined by (2.13) converge strongly to the same point $u_0 = P_{F(T)}(x_0)$*, as* $n \to \infty$.

Corolary 2.1 *Let* C *be a nonempty closed convex subset in a real Hilbert space* H *and let* $T : C \to H$ *be a nonexpansive mapping such that* $F(T) \neq \emptyset$. *Assume that* $\{\beta_n\}$ *is a sequence in [0,1] such that such that* $\beta_n \to 0$. *Then, the sequences* $\{x_n\}$ *and* $\{y_n\}$*, defined by*

$$\begin{cases} x_0 \in H \text{ any element,} \\ y_n = \beta_n x_0 + (1-\beta_n) P_C T P_C(x_n), \\ H_n = \{z \in H : \|y_n - z\|^2 \leq \|x_n - z\|^2 \\ \quad +\beta_n(\|x_0\|^2 + 2\langle x_n - x_0, z\rangle)\}, \\ W_n = \{z \in H : \langle x_n - z, x_0 - x_n\rangle \geq 0\}, \\ x_{n+1} = P_{H_n \cap W_n}(x_0), \quad n \geq 0, \end{cases}$$

converge strongly to the same point $u_0 = P_{F(T)}(x_0)$*, as* $n \to \infty$.

Corolary 2.2 *Let* C *be a nonempty closed convex subset in a real Hilbert space* H *and let* $T : C \to H$ *be a nonexpansive mapping such that* $F(T) \neq \emptyset$. *Assume that* $\{\alpha_n\}$ *is a sequence in [0,1] such that* $\alpha_n \to 1$. *Then, the sequences* $\{x_n\}$ *and* $\{y_n\}$*, defined by*

$$\begin{cases} x_0 \in H \text{ any element,} \\ y_n = P_C T(\alpha_n P_C(x_n) + (1-\alpha_n) P_C T P_C(x_n)), \\ H_n = \{z \in H : \|y_n - z\| \leq \|x_n - z\|\}, \\ W_n = \{z \in H : \langle x_n - z, x_0 - x_n\rangle \geq 0\}, \\ x_{n+1} = P_{H_n \cap W_n}(x_0), \quad n \geq 0, \end{cases}$$

converge strongly to the same point $u_0 = P_{F(T)}(x_0)$*, as* $n \to \infty$.

2.3. Hybrid Steepest Descent Methods

Sequence $\{x_n\}$ is defined by

$$\begin{cases} x_0 \in H = H_0, \\ y_n = x_n - \mu_n (I - TP_C)(x_n), \\ H_{n+1} = \{z \in H_n : \|y_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{H_{n+1}}(x_0), \quad n \geq 0. \end{cases} \tag{2.21}$$

We have the following result:

Theorem 2.4 *Let C be a nonempty closed convex subset in a real Hilbert space H and let T be a nonexpansive mapping on C such that $F(T) \neq \emptyset$. Assume that $\{\mu_n\}$ is a sequence in $(a, 1)$ for some $a \in (0, 1]$. Then, the sequences $\{x_n\}$ and $\{y_n\}$, defined by (2.21), converge strongly to the same point $u_0 = P_{F(T)}x_0$.*

2.4. Common Fixed Points For Two Nonexpansive Mappings On Two Subsets

Let C_1, C_2, be two closed and convex subsets in H and $T_1 : C_1 \to C_1$, $T_2 : C_2 \to C_2$ be two nonexpansive mapppings. Consider problem: Find

$$p \in F := F(T_1) \cap F(T_2), \tag{2.24}$$

with assumption that F is nonempty.

To solve problem (2.24) we propose the new method as follows:

$$\begin{cases} x_0 \in H \text{ any element,} \\ z_n = x_n - \mu_n(x_n - T_1 P_{C_1}(x_n)), \\ y_n = \beta_n x_0 + (1 - \beta_n) T_2 P_{C_2}(z_n), \\ H_n = \{z \in H : \|y_n - z\|^2 \leq \|x_n - z\|^2 \\ \qquad +\beta_n(\|x_0\|^2 + 2\langle x_n - x_0, z\rangle)\}, \\ W_n = \{z \in H : \langle x_n - z, x_0 - x_n\rangle \geq 0\}, \\ x_{n+1} = P_{H_n \cap W_n}(x_0), \quad n \geq 0. \end{cases} \tag{2.25}$$

We have the following theorem:

Theorem 2.5 *Let C_1 and C_2 be two nonempty, closed and convex subsets in a real Hilbert space H and let T_1 and T_2 be two nonexpansive mappings on C_1 and C_2, respectively, such that $F := F(T_1) \cap F(T_2) \neq \emptyset$. Assume that $\{\mu_n\}$ and $\{\beta_n\}$ are sequences in [0,1] such that $\mu_n \in (a, b)$ for some $a, b \in (0, 1)$ and $\beta_n \to 0$. Then, the sequences $\{x_n\}, \{z_n\}$ and $\{y_n\}$, defined by (2.25), converge strongly to the same point $u_0 = P_F(x_0)$, as $n \to \infty$.*

Corolary 2.3 *Let $C_i, i = 1, 2,$ be two nonempty, closed and convex subsets in a real Hilbert space H. Let $T_i, i = 1, 2,$ be two nonexpansive mappings on C_i such that $F(T_1) \cap F(T_2) \neq \emptyset$. Assume that $\{\mu_n\}$ is a sequence such that $0 < a \leq \mu_n \leq b < 1$. Then, the sequences $\{x_n\}$ and $\{y_n\}$, defined by*

$$\begin{cases} x_0 \in H \ \textit{any element}, \\ y_n = T_2 P_{C_2}(x_n - \mu_n(x_n - T_1 P_{C_1}(x_n))), \\ H_n = \{z \in H : \|y_n - z\| \leq \|x_n - z\|\}, \\ W_n = \{z \in H : \langle x_n - z, x_0 - x_n \rangle \geq 0\}, \\ x_{n+1} = P_{H_n \cap W_n}(x_0), \quad n \geq 0, \end{cases}$$

converge strongly to the same point $u_0 = P_{F(T)}(x_0)$, as $n \to \infty$.

Corolary 2.4 *Let $C_i, i = 1, 2,$ be two nonempty, closed and convex subsets in a real Hilbert space H such that $C := C_1 \cap C_2 \neq \emptyset$. Assume that $\{\mu_n\}$ and $\{\beta_n\}$ are sequences in [0,1] such that $\mu_n \in (a, b)$ for some $a, b \in (0, 1)$ and $\beta_n \to 0$. Then, the sequences $\{x_n\}, \{z_n\}$ and $\{y_n\}$, defined by*

$$\begin{cases} x_0 \in H \ \textit{any element}, \\ z_n = x_n - \mu_n(x_n - P_{C_1}(x_n)), \\ y_n = \beta_n x_0 + (1 - \beta_n) P_{C_2} z_n, \\ H_n = \{z \in H : \|y_n - z\|^2 \leq \|x_n - z\|^2 \\ \quad +\beta_n(\|x_0\|^2 + 2\langle x_n - x_0, z \rangle)\}, \\ W_n = \{z \in H : \langle x_n - z, x_0 - x_n \rangle \geq 0\}, \\ x_{n+1} = P_{H_n \cap W_n}(x_0), \quad n \geq 0, \end{cases}$$

converge strongly to the same point $u_0 = P_C(x_0)$, as $n \to \infty$.

2.5. Numerical Example

Example 2.1 Consider mapping T from $L_2[0, 1]$ into itself defined by

$$(T(x))(u) = 3 \int_0^1 usx(s)ds + 3u - 2, \tag{2.35}$$

for all $x \in L_2[0,1]$. Hence, T is a nonexpansive mapping.
Let f is a mapping from $L_2[0,1]$ into itself defined by

$$(f(x))(u) = \frac{1}{2}x(u), \text{ vi mi } x \in L_2[0,1]. \tag{2.36}$$

Then, f is a contraction with coefficient $\widetilde{\alpha} = \frac{1}{2}$.

Clearly, variational inequality: Find $p^* \in F(T)$ such that

$$\langle p^* - f(p^*), p - p^* \rangle \geq 0, \quad \forall p \in F(T), \tag{2.37}$$

has a unique solution $p^* = 3u - 2$.

From (2.1) we have

$$T^t = T_1^t T_0^t = T_1^t[(1-\lambda_t\mu)I + \lambda_t\mu f] = (1-\beta_t)(1-\frac{\lambda_t\mu}{2})I + \beta_t T((1-\frac{\lambda_t\mu}{2})I). \tag{2.38}$$

Choose $\beta_t = \beta = 10^{-4}$, $\mu = \frac{2}{5}$, $\lambda_t = \lambda = 10^{-4}$ and compute matrix

$$A = (1-(1-\beta)(1-\frac{\lambda\mu}{2}))I - 3\beta(1-\frac{\lambda\mu}{2})B$$

and right hand side $g = \beta(3u^T - (2,2,...,2)^T)$. Then, approximate solution is computed by formula $X = A^{-1}g$.

With exact solution $p^* = 3u - 2$.

Computing results at the iteration 20 are showed in the following table:
Table 2.1

Iteration u_i	App Solution $X(u_i)$	Exact Solution $p^*(u_i)$
$u_0 = 0.000000000000000$	-1.666694444908047	-2.000000000000000
$u_1 = 0.050000000000000$	-1.540906200737406	-1.850000000000000
....	....	....
$u_{20} = 1.000000000000000$	0.849070438504779	1.000000000000000

Next, we give computing result for explicit method (2.8).

Choose $\mu = \frac{2}{5}, \gamma_k = \frac{1}{2}, \lambda_k = \frac{1}{k}, \forall k \geq 1$ and use (2.8) we have
$X_{k+1} = (1-\gamma_k)X_k + \gamma_k(1-\frac{\lambda_k\mu}{2})(3BX_k + p)$.

Computing results at the iteration 20 are showed in the following table:

Table 2.2

Iteration u_i	App Solution $X(u_i)$	Exact Solution $p^*(u_i)$
$u_0 = 0.00000000000000$	-1.999998092651367	-2.00000000000000
$u_1 = 0.05000000000000$	-1.848447062448525	-1.85000000000000
....	....	....
$u_{20} = 1.000000000000000$	1.031022511405487	1.000000000000000

With the same problem, we consider the explicit iterations (2.9). We have $y_k = (1-\beta_k)x_k + \beta_k T x_k$ then, we have approximate equation $Y_k = (1-\beta_k)X_k + \beta_k(3BX_k + p)$, where

$$Y_k = (y_k(u_0), y_k(u_1), ..., y_k(u_M))^T, \ X_k = (x_k(u_0), x_k(u_1), ..., x_k(u_M))^T$$

and $p = 3(u_0, u_1, ..., u_M) - (2, 2, ..., 2)$.
Choose $\mu = \frac{2}{5}$, $\beta_k = \gamma_k = \frac{1}{2}$, $\lambda_k = \frac{1}{k}$ for all $k \geq 1$, by using (2.9) we have $X_{k+1} = (1-\gamma_k)X_k + \gamma_k(1 - \frac{\lambda_k \mu}{2})Y_k$.

Computing result at the 50th iteration is showed in the following table.
Table 2.3

Iteration u_i	App Solution $X(u_i)$	Exact Solution $p^*(u_i)$
$u_0 = 0.00000000000000$	-1.982945017736413	-2.00000000000000
$u_1 = 0.05000000000000$	-1.832285258509282	-1.85000000000000
....	....	....
$u_{20} = 1.000000000000000$	0.849070438504779	1.000000000000000

Example 2.2 In $\mathbb{R}^2$, let S_1 and S_2 be two circles defined by

$$S_1: \ (x-2)^2 + (y-2)^2 \leq 1, \ S_2: \ (x-4)^2 + (y-2)^2 \leq 4.$$

Consider the problem of finding x^*, such that $x^* \in S = S_1 \cap S_2$.
By the same argument, we choose $\alpha_n = 1 - \frac{1}{n+1}$, $\beta_n = \frac{1}{n}$, $x_0 = \left(\frac{9}{4}, 0\right)$ and compute $x_{n+1} = P_{H_n \cap W_n}(x_0)$.

Computing results at the 1000th iteration is showed in the following table.

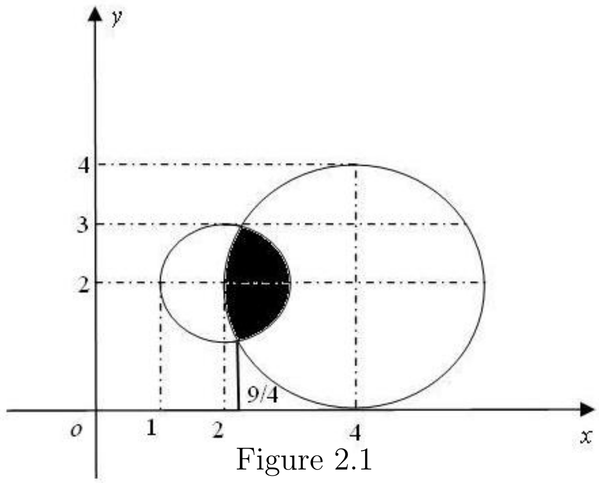

Figure 2.1

Table 2.4

Solution		App Solution x_n		App Solution y_n		App Solution z_n	
x^1	x^2	x_n^1	x_n^2	y_n^1	y_n^2	z_n^1	z_n^2
2.2500000	1.0317541	2.2332447	1.0319233	2.2396581	1.0343974	2.2332510	1.03192782

Example 2.3 In $\mathbb{R}^2$, let C_1 and C_2 be two subsets defined by

$$C_1 = \{(x, y) \in \mathbb{R}^2 : \ 0 \leq x, \ y \leq 1\},$$
$$C_2 = \{(x, y) \in \mathbb{R}^2 : \ 3x - 2y \geq -1, \ x + 4y \geq 2, \ 2x + y \leq 4\}.$$

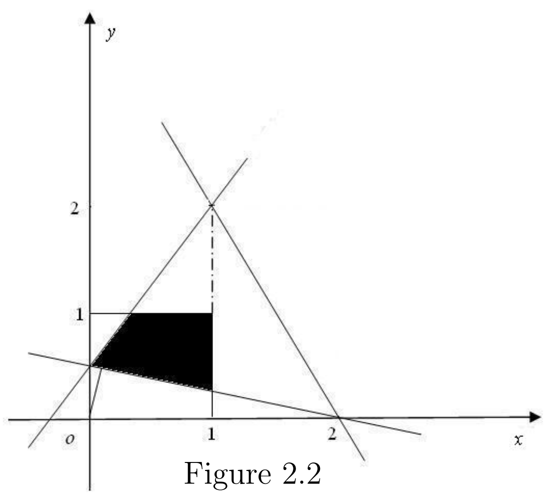

Figure 2.2

The computation of super plane H_n, W_n and projection of x_0 onto H_n, W_n is established the same as in Example 2.2.

Choose $x_0 = (0, 0)$, $\beta_n = \dfrac{1}{n}$, $\mu_n = \dfrac{1}{2}$, compute $x_{n+1} = P_{H_n \cap W_n}(x_0)$.

Computing results at the 5000th iteration is showed in the following table.

Table 2.5

Solution		x_n		y_n		z_n	
x^1	x^2	x_n^1	x_n^2	y_n^1	y_n^2	z_n^1	z_n^2
0.1176470	0.4705882	0.1153171	0.4612687	0.1176235	0.4704941	0.1153169	0.4612678

Example 2.4 Consider the problem of finding a common point of two circles as in Example 2.2, with the iteration $\{x_n\}$ defined by (2.21). Choose $x_0 = \big(\frac{9}{4}, 0\big)$, $\mu_n = \frac{1}{2}$ and compute

$$x_{n+1} = P_{H_{n+1}}(x_0) = P_{W_0 \cap W_1 \dots \cap W_n}(x_0).$$

Then, to determine $P_{H_{n+1}}(x_0)$, we can use the cyclic projection method in the form

$$u_{k+1} = P_{W_{k \bmod n}}(u_k),\ u_0 = x_0,\ k \geq 0,$$

or the following iterative method

$$u_{k+1} = \frac{\sum_{i=1}^{n} P_{W_i}(u_k)}{n},\ u_0 = x_0,\ k \geq 0. \tag{2.41}$$

Now we use the iterative method (2.41) to compute approximation of $P_{H_{n+1}}(x_0)$.

Computing results at the 200th iteration is showed in the following table.
Table 2.6

Solution		x_n		y_n	
x^1	x^2	x_n^1	x_n^2	y_n^1	y_n^2
2.2500000000	1.0317541634	2.2499871121	1.0317755681	2.2500564711	1.0317684570

Remark 2.1 Based on the computing results for the considered iteration methods showed in these above tables, we can conclude that the larger iteration is the closer exact solution of approximate one is.

Chapter 3

Approximative Methods For Fixed Points of Nonexpansive Semigroups

3.1. Common Fixed Points of Nonexpansive Semigroups

To find an element $p \in \mathcal{F}$, based on Mann iteration, Halpern iteration and hybrid steepest descent methods using in mathematical programming, we propose a new iterative method as follows:

$$\begin{cases} x_0 \in H \text{ any element}, \\ z_n = \alpha_n P_C(x_n) + (1-\alpha_n)\frac{1}{t_n}\int_0^{t_n} T(s)P_C(x_n)ds, \\ y_n = \beta_n x_0 + (1-\beta_n)\frac{1}{t_n}\int_0^{t_n} T(s)z_n ds, \\ H_n = \{z \in H : \|y_n - z\|^2 \le \|x_n - z\|^2 \\ \qquad +\beta_n(\|x_0\|^2 + 2\langle x_n - x_0, z\rangle)\}, \\ W_n = \{z \in H : \langle x_n - z, x_0 - x_n\rangle \ge 0\}, \\ x_{n+1} = P_{H_n \cap W_n}(x_0), \quad n \ge 0, \end{cases} \tag{3.1}$$

for a nonexpansive semigroup on C.
We will give strong convergence of the iterative sequences $\{x_n\}, \{y_n\}$ and $\{z_n\}$ defined by (3.1) to a common fixed point of nonexpansive semigroup $\{T(t) : t \ge 0\}$ with some certain conditions imposed on parameters $\{\alpha_n\}$, $\{\beta_n\}$, and $\{t_n\}$.

Theorem 3.1 *Let C be a nonempty closed convex subset in a real Hilbert space H and let $\{T(t) : t \ge 0\}$ be a nonexpansive semigroup on C such that $\mathcal{F} = \cap_{t\ge 0} F(T(t)) \ne \emptyset$. Assume that $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in [0,1] such that $\alpha_n \to 1$ and $\beta_n \to 0$, and $\{t_n\}$ is a* *positive real divergent sequence. Then, the sequences* $\{x_n\}, \{z_n\}$ *and* $\{y_n\}$*, defined by* (3.1)*, converge strongly to the same point* $u_0 = P_{\mathcal{F}}(x_0)$*, as* $n \to \infty$*.*

Corolary 3.1 *Let* C *be a nonempty closed convex subset in a real Hilbert space* H *and let* $\{T(t) : t \geq 0\}$ *be a nonexpansive semigroup on* C *such that* $\mathcal{F} = \cap_{t\geq 0} F(T(t)) \neq \emptyset$*. Assume that* $\{\beta_n\}$ *is a sequence in [0,1] such that* $\beta_n \to 0$*. Then, the sequences* $\{x_n\}$ *and* $\{y_n\}$*, defined by*

$$\begin{cases} x_0 \in H \text{ any element,} \\ y_n = \beta_n x_0 + (1-\beta_n)\frac{1}{t_n}\int_0^{t_n} T(s)P_C(x_n)ds, \\ H_n = \{z \in H : \|y_n - z\|^2 \leq \|x_n - z\|^2 \\ \quad +\beta_n(\|x_0\|^2 + 2\langle x_n - x_0, z\rangle)\}, \\ W_n = \{z \in H : \langle x_n - z, x_0 - x_n\rangle \geq 0\}, \\ x_{n+1} = P_{H_n \cap W_n}(x_0), \quad n \geq 0, \end{cases}$$

converge strongly to the same point $u_0 = P_{\mathcal{F}}(x_0)$*, as* $n \to \infty$*.*

Corolary 3.2 *Let* C *be a nonempty closed convex subset in a real Hilbert space* H *and let* $\{T(t) : t \geq 0\}$ *be a nonexpansive semigroup on* C *such that* $\mathcal{F} = \cap_{t\geq 0} F(T(t)) \neq \emptyset$*. Assume that* $\{\alpha_n\}$ *is a sequence in [0,1] such that* $\alpha_n \to 1$*. Then, the sequences* $\{x_n\}$ *and* $\{y_n\}$*, defined by*

$$\begin{cases} x_0 \in H \text{ any element,} \\ y_n = \frac{1}{t_n}\int_0^{t_n} T(s)\left[\alpha_n P_C(x_n) + (1-\alpha_n)\frac{1}{t_n}\int_0^{t_n} T(s)P_C(x_n)ds\right]ds, \\ H_n = \{z \in H : \|y_n - z\| \leq \|x_n - z\|\}, \\ W_n = \{z \in H : \langle x_n - z, x_0 - x_n\rangle \geq 0\}, \\ x_{n+1} = P_{H_n \cap W_n}(x_0), \quad n \geq 0, \end{cases}$$

converge strongly to the same point $u_0 = P_{\mathcal{F}}(x_0)$*, as* $n \to \infty$*.*

Next, we prgive an improvement of hybrid steepest descent method for the problem of finding an element $p \in \mathcal{F}$. To be specific, we consider the following method:

$$\begin{cases} x_0 \in H = H_0, \\ y_n = x_n - \mu_n(I - T_n P_C)(x_n), \\ H_{n+1} = \{z \in H_n : \|y_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{H_{n+1}}(x_0), \quad n \geq 0 \end{cases} \tag{3.9}$$

and

$$\begin{cases} x_0 \in H = H_0, \\ y_n = x_n - \mu_n(I - T(t_n)P_C(x_n)), \\ H_{n+1} = \{z \in H_n : \|y_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{H_{n+1}}(x_0), \quad n \geq 0. \end{cases} \tag{3.10}$$

The strong convergence of (3.9) is stated in the following theorem:

Theorem 3.2 *Let C be a nonempty closed convex subset in a real Hilbert space H and let $\{T(t) : t \geq 0\}$ be a nonexpansive semigroup on C such that $\mathcal{F} = \cap_{t \geq 0} F(T(t)) \neq \emptyset$. Assume that $\{\mu_n\}$ is a sequence in $(a, 1]$ for some $a \in (0, 1]$ and $\{\lambda_n\}$ is a positive real number divergent sequence. Then, the sequences $\{x_n\}$ and $\{y_n\}$ defined by (3.9), converge strongly to the same point $u_0 = P_{\mathcal{F}}(x_0)$.*

Next, the strong convergence of method (3.10) is given in the following theorem:

Theorem 3.3 *Let C be a nonempty closed convex subset in a real Hilbert space H and let $\{T(t) : t \geq 0\}$ be a nonexpansive semigroup on C such that $\mathcal{F} = \cap_{t \geq 0} F(T(t)) \neq \emptyset$. Assume that $\{\mu_n\}$ is a sequence in $(a, 1]$ for some $a \in (0, 1]$ and $\{t_n\}$ is a sequence of positive real numbers satisfying the condition $\liminf_n t_n = 0, \limsup_n t_n > 0$, and $\lim_n (t_{n+1} - t_n) = 0$. Then, the sequences $\{x_n\}$ and $\{y_n\}$ defined by (3.10), converge strongly to the same point $u_0 = P_{\mathcal{F}}(x_0)$.*

3.2. Common Fixed Point of Two Nonexpansive Semigroups

Let $C_1,\ C_2$ be two closed and convex subsets in Hilbert space H and $\{T_1(t) : t \geq 0\}, \{T_2(t) : t \geq 0\}$ be two nonexpansive semigroups from

C_1, C_2 into itself, respectively. The problem considered in this section is: Finding

$$q \in \mathcal{F}_{1,2} := \mathcal{F}_1 \cap \mathcal{F}_2, \tag{3.17}$$

when $\mathcal{F}_i = \cap_{t>0} F(T_i(t))$. ($\mathcal{F}_1, \mathcal{F}_2$ is nonempty).

Based on (3.17) we give a new iterative method

$$\begin{cases} x_0 \in H \text{ any element}, \\ z_n = x_n - \mu_n\Big(x_n - \frac{1}{t_n}\int_0^{t_n} T_1(s)P_{C_1}(x_n)ds\Big), \\ y_n = \beta_n x_0 + (1-\beta_n)\frac{1}{t_n}\int_0^{t_n} T_2(s)P_{C_2}(z_n)ds, \\ H_n = \{z \in H : \|y_n - z\|^2 \leq \|x_n - z\|^2 \\ \qquad +\beta_n(\|x_0\|^2 + 2\langle x_n - x_0, z\rangle)\}, \\ W_n = \{z \in H : \langle x_n - z, x_0 - x_n\rangle \geq 0\}, \\ x_{n+1} = P_{H_n \cap W_n}(x_0), \quad n \geq 0, \end{cases} \tag{3.18}$$

and prove the strong convergence of sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ defined by (3.18) to an element $q = u_0 \in \mathcal{F}_{1,2}$.

Theorem 3.4 *Let C_1 and C_2 be two nonempty closed convex subsets in a real Hilbert space H and let $\{T_1(t) : t \geq 0\}$ and $\{T_2(t) : t \geq 0\}$ be two nonexpansive semigroups on C_1 and C_2, respectively, such that $\mathcal{F} = \mathcal{F}_1 \cap \mathcal{F}_2 \neq \emptyset$ where $\mathcal{F}_i = \cap_{t>0} F(T_i(t)), i = 1, 2$. Assume that $\{\mu_n\}$ and $\{\beta_n\}$ are sequences in [0,1] such that $\mu_n \in (a, b)$ for some $a, b \in (0, 1)$ and $\beta_n \to 0$ and $\{t_n\}$ is a positive real divergent sequence. Then, the sequences $\{x_n\}, \{z_n\}$ and $\{y_n\}$, defined by (3.18), converge strongly to the same point $u_0 = P_{\mathcal{F}}(x_0)$, as $n \to \infty$.*

Corolary 3.3 *Let C be a nonempty closed convex subset in a real Hilbert space H and let $\{T(t) : t \geq 0\}$ be a nonexpansive semigroup on C such that $\mathcal{F} = \cap_{t\geq 0} F(T(t)) \neq \emptyset$. Assume that $\{\beta_n\}$ is a sequence in [0,1] such that $\beta_n \to 0$. Then, the sequences $\{x_n\}$ and $\{y_n\}$, defined by*

$$\begin{cases} x_0 \in H \text{ any element}, \\ y_n = \beta_n x_0 + (1-\beta_n)\frac{1}{t_n}\int_0^{t_n} T(s)P_C(x_n)ds, \end{cases}$$

$$\begin{cases} H_n = \{z \in H : \|y_n - z\|^2 \leq \|x_n - z\|^2 \\ \quad +\beta_n(\|x_0\|^2 + 2\langle x_n - x_0, z\rangle)\}, \\ W_n = \{z \in H : \langle x_n - z, x_0 - x_n\rangle \geq 0\}, \\ x_{n+1} = P_{H_n \cap W_n}(x_0), \quad n \geq 0, \end{cases}$$

converge strongly to the same point $u_0 = P_{\mathcal{F}}(x_0)$, *as* $n \to \infty$.

Corolary 3.4 *Let* C *be a nonempty closed convex subset in a real Hilbert space* H *and let* $\{T(t) : t > 0\}$ *be a nonexpansive semigroup on* C *such that* $\mathcal{F} = \cap_{t>0} F(T(t)) \neq \emptyset$. *Assume that* $\{\alpha_n\}$ *is a sequence in [0,1] such that* $\alpha_n \to 1$. *Then, the sequences* $\{x_n\}$ *and* $\{y_n\}$, *defined by*

$$\begin{cases} x_0 \in H \text{ any element}, \\ y_n = \frac{1}{t_n}\int_0^{t_n} T(s)P_C\left(x_n - \mu_n\left[x_n - \frac{1}{t_n}\int_0^{t_n} T(s)P_C x_n ds\right] ds\right), \\ H_n = \{z \in H : \|y_n - z\| \leq \|x_n - z\|\}, \\ W_n = \{z \in H : \langle x_n - z, x_0 - x_n\rangle \geq 0\}, \\ x_{n+1} = P_{H_n \cap W_n}(x_0), \quad n \geq 0, \end{cases}$$

converge strongly to the same point $u_0 = P_{\mathcal{F}}(x_0)$, *as* $n \to \infty$.

3.3. Numerical Example

Example 3.1 In $\mathbb{R}^2$, with $t > 0$, consider mappings $T(t) : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(t)x = \begin{pmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, for all $x = (x_1, x_2) \in \mathbb{R}^2$.

Choose $x_0 = (-1, 1)$, $\alpha_n = 1 - \dfrac{1}{n+1}$, $\beta_n = \dfrac{1}{n}$, $t_n = n\pi$ and compute $x_{n+1} = P_{H_n \cap W_n}(x_0)$. The computation of hyper planes H_n, W_n and projection of x_0 onto H_n, W_n is the same as in Example 2.2.

Computing results at the 500th iteration is showed in the following table.
Table 3.1

Solution		x_n		y_n		z_n	
x^1	x^2	x_n^1	x_n^2	y_n^1	y_n^2	z_n^1	z_n^2
0	0	-0.031259	-0.031259	-0.014563	-0.014563	-0.031230	-0.031230

Besides, the convergences of sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ to solution $(0, 0)$ are showed in the following figure.

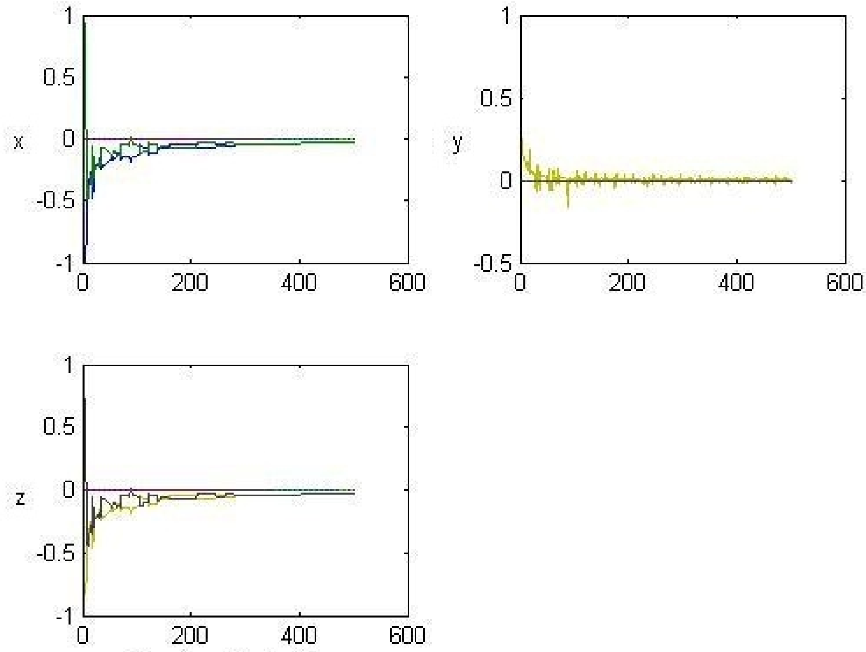

Figure 3.1

Then we can compute $y_n = (1-\mu_n)x_n + \mu_n T_n P_C(x_n)$ and the computation of H_{n+1}, W_n and $P_{H_{n+1}}(x_0)$ is the same as in Example 2.4.

Choose $x_0 = (-1, 1)$, $\mu_n = \dfrac{1}{2}$, $t_n = n\pi$.

Computing results at the 50th iteration is showed in the following table.

Table 3.2

Solution		x_n		y_n	
x^1	x^2	x_n^1	x_n^2	y_n^1	y_n^2
0	0	-0.735×10^{-3}	0.445×10^{-3}	0.461×10^{-3}	-0.239×10^{-3}

Computing results at the 50th iteration is also showed in figure as follows.

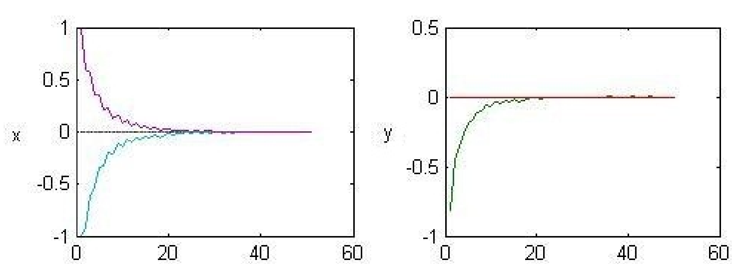

Figure 3.2

Example 3.2 In this example, consider iterative method (3.18) for solving the problem of finding common fixed points of two nonexpansive semigroups $\{T_m(t)\}$ defined by $\begin{pmatrix} \cos(mt) & -\sin(mt) \\ \sin(mt) & \cos(mt) \end{pmatrix}$, $m = 1, 2.$

Choose $x_0 = (-1, 1), \mu_n = \frac{1}{2}, \beta_n = \frac{1}{n}, t_n = n\pi$ and compute $x_{n+1} = P_{H_n \cap W_n}(x_0)$, where the computation of $H_n,\ W_n$ and projection of x_0 onto $H_n,\ W_n$ is the same as in Example 2.2.

Computing results at the 500th iteration is showed in the following table.
Table 3.3

Solution		x_n		y_n		z_n	
x^1	x^2	x_n^1	x_n^2	y_n^1	y_n^2	z_n^1	z_n^2
0	0	-0.036923	-0.037136	-0.008730	-0.008784	-0.027451	-0.027611

The strong convergence of the above method is also illustrated in the following figure.

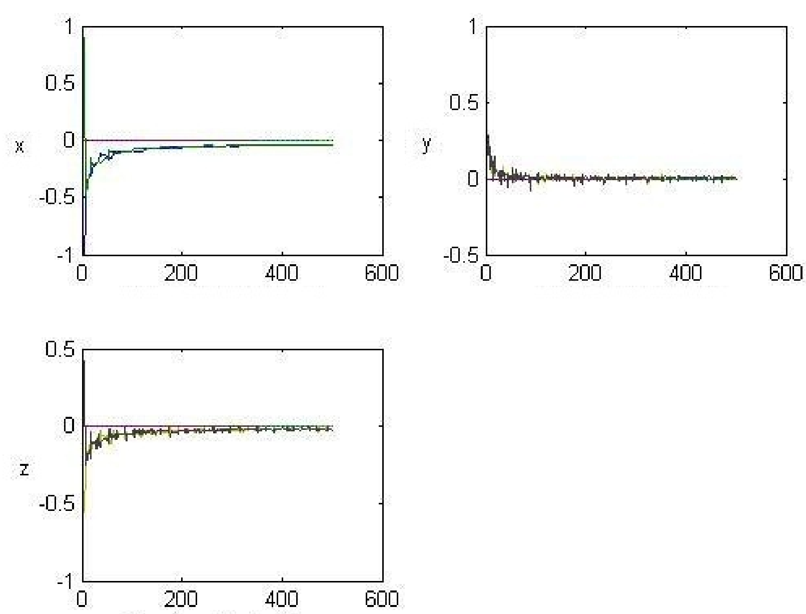

Figure 3.3

Remark 3.1 From the tables of computing results for considered iterative methods we can conclude that if the iteration is higher and higher then the approximate solutions are closer to exact solution.

Final conclusion and further recommendation

Thesis has mentioned the following issues.

1. Study an improvement of Moudafi's result in order to obtain the strong convergence of implicit and explicit methods with "milder" conditions imposed on parameters. We also combined Mann iteration method, Halpern iteration, and hybrid steepest descent method in mathematical programming for finding common fixed points of a nonexpansive mapping on a closed and convex subset C or common fixed points of two nonexpansive mappings on two closed and convex subsets with nonempty intersection in Hilbert spaces H. The strong convergence of hybrid steepest descent methods to common fixed point of a nonexpansive mapping is proved.

2. Consider combination of Mann iteration method, Halpern iteration, and hybrid steepest descent method in mathematical programming for finding common fixed points of nonexpansive semigroup on a closed and convex subset C or common fixed points of two nonexpansive semigroups on two closed and convex subsets with nonempty intersection in Hilbert spaces H. We also studied the strong convergence of hybrid steepest descent method for the problem of finding common fixed points of nonexpansive semigroups.

Recommend futher research

1. Use the results, obtained in our thesis, to solve more complicated problems.

2. Extension of the results from Hilbert spaces to Banach spaces.

The list of published works related to thesis

(1). Nguyen Buong, Nguyen Duc Lang (2011), "Shrinking hybrid descent-like methods for nonexpansive mappings and semigroups", *Nonlinear Functional Analysis and Applications.*, Vol. 16, No. 3, pp. 331-339.

(2). Nguyen Buong, Nguyen Duc Lang (2011), "Iteration methods for fixed point of a nonexpansive mapping", *International Mathematical Forum.*, Vol. 6, No. 60, pp. 2963-2974.

(3). Nguyen Buong, Nguyen Duc Lang (2011), "Hybrid Mann - Halpern iteration methods for nonexpansive mappings and semigroups", *Applied Mathematics and Computation.*, Vol. 218, Issue 6, pp. 2459-2466.

(4). Nguyen Buong, Nguyen Duc Lang (2012), "Hybrid descent - like halpern iteration methods for two nonexpansive mappings and semigroups on two sets", *Theoretical Mathematics & Applications.*, Vol. 2, No. 3, pp. 23-38.